SEMICLASSICAL EXPANSIONS UP TO $\hbar^4$-ORDER IN RELATIVISTIC NUCLEAR PHYSICS

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ABSTRACT

We present the first calculation of the $\hbar^4$-Wigner–Kirkwood corrections to a relativistic system of fermions in the presence of external scalar and vector potentials. The method we propose allows to compute efficiently semiclassical corrections to one body operators such as mean energies and local densities. It also preserves gauge invariance and produces explicitly convergent results despite some apparent divergencies at the classical turning points. As a byproduct we obtain the $\hbar^4$ corrections stemming from the polarization of the Dirac sea. We apply our results to the relativistic σ-ω Lagrangian in the Hartree valence approximation. We compare the semiclassical expansion with the exact result and with the Strutinsky average whenever it can be obtained. We find that the $\hbar^4$ corrections are much smaller than typical shell effects. Our results provide convincing arguments to neglect higher than second order $\hbar$ effects in the Wigner–Kirkwood scheme to relativistic nuclear physics in the mean field approximation.

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1   Introduction

Relativistic mean field theories for nuclear systems have already a long history [1] and have found a wealth of applications [2, 3, 4]. They use as a starting point a relativistic quantum field Lagrangian which hopefully includes the relevant degrees of freedom, i.e. nucleons and several mesons. In the Hartree approximation it is possible to reproduce the saturation properties for nuclear matter and the shell structure of single particle states is accurately described due to the correct sign and size of the spin-orbit interaction. Many of the calculations are performed in practice for spherical closed shell nuclei, where shell effects are expected to be least significant. For these systems many authors have proposed to employ a relativistic Thomas-Fermi (TF) model [2] and generalizations thereof [5, 6, 7, 8, 9, 10, 11, 12, 13].

From a purely computational point of view, the study of the semiclassical expansion for relativistic systems is far from being a trivial generalization of well-known non-relativistic methods which have been discussed at length for nuclear systems in the literature (for reviews see e.g. [14, 15] and references therein). The main problem is that whereas in the Schrödinger equation the Planck's constant $\hbar$ appears in the second power, in the Dirac equation $\hbar$ appears in the first power. The final results for bulk properties do not depend, however, on the sign of $\hbar$ due to parity invariance. This simple circumstance makes the direct generalization of non-relativistic methods such as e.g. Wigner transformation or others quite time consuming and generate extremely complicated expressions in intermediate stages of the calculation. An additional difficulty in the relativistic case is the appearance of four dimensional spinor matrices. One should also say that the treatment of nuclei requires at least a scalar potential in addition to the usual electrostatic potential of atomic physics, therefore making the computations much more involved. This probably explains why until now only the corrections up to $\hbar^2$ have been computed in relativistic nuclear systems [5, 6] and up to $\hbar^4$ in atomic systems [16]. In this respect one should mention that in the non-relativistic case the $\hbar^4$, and $\hbar^6$ have been known since a long time [17, 18] and quite recently $\hbar^8$-order corrections have been determined [19]. The question arises whether there are reasons to neglect higher orders than $\hbar^2$, besides the desire to keep the problem at a manageable level.

The answer to this question cannot be given in a closed form, because in general it is not known whether the semiclassical series is convergent or only asymptotic. In the absence of exact theorems only numerical investigation remains. As a practical rule there are cases like e.g. the non-relativistic harmonic oscillator where the Strutinsky average is known to reproduce the complete Wigner-Kirkwood (WK) result [20]. This rule seems also to be fulfilled for the relativistic harmonic potential [11] in order

WK$\hbar^2$.[1] However, in the cases of relevance for nuclear physics, i.e. short range and finite depth potentials, the Strutinsky average is in general not well defined, although nothing prevents from an order by order semiclassical calculation.

In the present paper we propose an efficient computational method which allows to compute up to WK$\hbar^4$-corrections of the semiclassical expansion for one-body operators following the method proposed in previous work [12]. This method is fully equivalent to the standard WK expansion [21, 22, 23] for one body operators; the main advantage is that one does not need to evaluate non-diagonal elements of the Green's functions and traces back all physical quantities to the cumulative number of states. As an aside we compute analytically the WK$\hbar^4$-corrections coming from the polarization of the Dirac sea. We apply our results to the relativistic σ-ω model in the Hartree valence approximation and make a numerical estimate of the WK$\hbar^4$-corrections to the fermionic contribution to the total energy. To set in the proper scales we adjust the parameters to reproduce the nuclear matter saturation properties and the charge mean squared radius of ^{40}Ca [24]. In doing so we do not pretend to make a full account of finite nuclei properties, since there are more realistic generalizations of this model to include Coulomb repulsion, tensor force, etc. [2, 3, 4]. Nevertheless we feel that the general features obtained in the present study should not be substantially modified by these extensions of the σ-ω model.

The outline of the paper is as follows. In section 2 we review the properties of a D-dimensional non-relativistic case to illustrate the basic mathematical objects such as the determinant of the Schrödinger operator and our method to perform semiclassical expansions in the simplest situation. As an example we apply our formalism to the three dimensional harmonic oscillator. We also discuss how a re-ordering of the semiclassical series proposed recently [25, 26] can be incorporated into the formalism, proving extremely convenient for numerical computations. Section 3 studies the relativistic case and the particular features of the determinant of the Dirac equation and explicit calculations up to $\hbar^4$-order are also presented. As an illustration we apply and discuss those corrections in the analytically solvable example of relativistic harmonic potentials with special emphasis on the distributional character of the quantities involved. We also show an explicit calculation of the (finite) WK$\hbar^4$-corrections arising from the polarization of the Dirac sea. In section 4 we quickly review the σ-ω model together with the parameter fixing and give numerical estimates to the WK$\hbar^4$-corrections to the Hartree energy. We also provide a prescription how to avoid the distributions in practical calculations. Finally, in section 5 we draw our conclusions and present some perspectives for future research.

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[1]The notation emphasizes that we are performing a WK expansion throughout, rather than an extended TF, that is, we expand in gradients of the potentials instead of gradients of the densities.

2 The Number of States Formalism

2.1 *Cumulative Number of States*

In this work we will consider objects which involve single sums, i.e. one body operators. All the information concerning them can be deduced from the single particle level density. As it will become clear below, it is more convenient to deal with the cumulative number of states. Let us consider the eigenvalue problem

$$H \, \psi_{n\alpha}(x) = E_n \, \psi_{n\alpha}(x) \tag{1}$$

where H is the single particle Hamiltonian, E_n are the energy eigenvalues, possibly g_n-fold degenerated, and $\psi_{n\alpha}(x)$ are the corresponding eigenfunctions ($\alpha = 1, \ldots, g_n$). To fix ideas we will assume the Hamiltonian to be of the Schrödinger form

$$H = -\frac{\hbar^2}{2m} \mathcal{D}^2 + V(x) \tag{2}$$

although many of the features presented here can be generalized. Here $\mathcal{D}$ represents a covariant space derivative $\mathcal{D}_i = \nabla_i + \frac{i}{\hbar} A_i$ with A_i the vector gauge potential which includes the coupling constant. The cumulative number of states $N(E)$, counting the number of states below a certain energy E, is defined as

$$N(E) = \sum_{n=0}^{\infty} g_n \, \Theta(E - E_n) = \text{Tr} \, \Theta(E - H), \tag{3}$$

where g_n stands for the possible degeneracy of the system. From here the level density can be derived to be

$$g(E) = \frac{d}{dE} N(E) = \sum_{n=0}^{\infty} g_n \, \delta(E - E_n). \tag{4}$$

Both $N(E)$ and $g(E)$ are to be interpreted as distributions. For our purposes it is extremely convenient to consider the following representation of the step function distribution

$$\Theta(x) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \, \text{Im} \log(-x + i \, \epsilon) = \begin{cases} 1 & \text{if } x > 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{5}$$

the limit $\epsilon \to 0^+$ being always understood at the end of the calculations. The branch cut for the log function is considered to be along the positive real axis as follows

$$\log z = \log |z| + i \, \arg z, \qquad 0 \le \arg z < 2\pi, \tag{6}$$

4

which has the property

$$\frac{1}{2} \log \left[(-x + i\epsilon)^2\right] = \log(-x + i\epsilon).$$ (7)

This property will be used later in the relativistic case. With these definitions the cumulative number of states can be written as

$$N(E) = \frac{1}{\pi} \operatorname{Im} \operatorname{Tr} \log\left(H - E + i\,\epsilon\right).$$ (8)

If we further use the identity

$$\operatorname{Tr} \log A = \log \det A$$ (9)

with A a linear operator, the following formula is obtained

$$N(E) = \frac{1}{\pi} \arg \det(H - E + i\,\epsilon),$$ (10)

i.e. the cumulative number of states is proportional to the phase of the determinant of the operator $H - E + i\,\epsilon$. It should be mentioned that the definition of $N(E)$ implicitly assumes that the Hamiltonian is bounded from below. If this is not the case one can still proceed with the aforementioned formulas but subtracting a suitable infinite but energy independent constant. This point will be discussed when dealing with the Dirac equation.

2.2 *Physical Quantities*

Any single sum can be obtained directly from $N(E)$. For instance, the total number of particles below a chemical potential μ is given by $N(E = \mu)$. Similarly, the total energy of a system of $N(\mu)$ fermions can be expressed as

$$E_T(\mu) = \sum_{n=0}^{\infty} g_n \, E_n \, \Theta(\mu - E_n) = \int dE \, g(E) \, E \, \Theta(\mu - E).$$ (11)

In this particular case integration by parts yields the simpler result

$$E_T(\mu) = \mu N(\mu) - \int dE \, N(E) \, \Theta(\mu - E),$$ (12)

which is particularly interesting when evaluating the free energy. One body local quantities, such as particle densities and energy densities can be obtained from suitable functional derivatives with respect to external potentials with proper quantum

numbers. In the non-relativistic case[2] the particle density of a system of $N(\mu)$ fermions is

$$\rho(x, \mu) = \sum_{n=0}^{\infty} \sum_{\alpha=1}^{g_n} \psi_{n\alpha}^*(x)\, \psi_{n\alpha}(x)\, \Theta(\mu - E_n). \tag{13}$$

Considering the variation of $N(E)$ in eq. (3) under a variation $\delta V(x)$ of the potential

$$\delta N(E) = -\operatorname{Tr}\left(\delta(E - H)\delta V(x)\right) \tag{14}$$

one readily obtains the following useful result[3]

$$\frac{\delta N(E)}{\delta V(x)} = -\sum_{n=0}^{\infty} \sum_{\alpha=1}^{g_n} \psi_{n\alpha}^*(x)\, \psi_{n\alpha}(x)\, \delta(E - E_n), \tag{15}$$

yielding

$$\rho(x, \mu) = -\int dE\, \frac{\delta N(E)}{\delta V(x)}\, \Theta(\mu - E). \tag{16}$$

The total energy density

$$\rho_E(x, \mu) = \sum_{n=0}^{\infty} \sum_{\alpha=1}^{g_n} \psi_{n\alpha}^*(x)\, E_n\, \psi_{n\alpha}(x)\, \Theta(\mu - E_n), \tag{17}$$

can be obtained in a similar way,

$$\rho_E(x, \mu) = -\int dE\, \frac{\delta N(E)}{\delta V(x)}\, E\, \Theta(\mu - E). \tag{18}$$

For the kinetic energy density (total minus potential) one has

$$\rho_K(x, \mu) = -\int dE\, \frac{\delta N(E)}{\delta V(x)}\, (E - V(x))\, \Theta(\mu - E). \tag{19}$$

Other local densities can be also computed by introducing an external potential with suitable quantum numbers, for instance the current density could be evaluated by a functional derivative with respect to the external vector potential.

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[2] The results for relativistic systems will be presented in section 3.

[3] Note that the result is base independent due to the sum in the eigensubspace.

2.3 Semiclassical Expansion in the Non-Relativistic case.

As we have said any one body operator can be deduced from the cumulative number of states which in our regularization turns out to be proportional to the phase of the determinant of $H - E + i\epsilon$. Our aim is to perform a semiclassical expansion of those objects. Besides an overall factor this expansion correspond to a derivative expansion in the potential $V(x)$. Therefore in most of the expressions the $\hbar$-dependence can be directly read off. To perform this expansion we use the method of Chan [27] recently extended by two of us [28] and also considered in previous work [12] in the present context. In the appendix we review for completeness the main features of the method and provide some results which will be extensively used along this paper.

Many of the features which occur in higher order WK$\hbar$ corrections within our formalism are best exemplified in the non-relativistic case. Thus, let us consider the semiclassical expansion of the cumulative number of states for a system of non-relativistic fermions with spinor degeneracy ν interacting with a spin independent mean field potential (the generalization to spin dependent forces is straightforward although somewhat more involved). This case is well known from the study of non-relativistic atomic systems in the context of density functional theory [29] and we only show it here for pedagogical reasons. We will do this up to fourth order in $\hbar$ in D-dimensions. This particular example is also interesting because it illustrates, in the case $D = 3$, the meaning of apparently divergent integrals. This feature does not show up at orders in $\hbar$ strictly lower than four. As shown in the appendix it is convenient to consider the function Δ given by

$$\Delta(x, k) \equiv \frac{1}{k^2 + 2m(V(x) - E) + i\epsilon} \,. \tag{20}$$

Furthermore we use the notation (see caption of table 9)

$$\begin{aligned}
\Delta_i &= \hbar \nabla_i \Delta \\
F_{ij} &= \hbar (\nabla_i A_j - \nabla_j A_i) \\
F^2 &\equiv F_{ij} F_{ij} \qquad i, j = 1, 2, \ldots, D \,,
\end{aligned} \tag{21}$$

where F_{ij} represents the field strength tensor of the vector gauge potential A_i times $\hbar$. In this particular case Δ possesses a trivial matrix structure, which simplifies a lot the actual calculation. One has after application of table 9 the result

$$N(E) = N_0(E) + N_2(E) + N_4(E) + \ldots , \tag{22}$$

where

$$N_0(E) = \frac{1}{\pi} \operatorname{Im} \int \frac{d^D x \, d^D k}{(2\pi\hbar)^D} \log(k^2 + 2m(V(x) - E) + i\epsilon)$$

$$N_2(E) = \frac{1}{\pi} \frac{\hbar^2}{D} \, \text{Im} \int \frac{d^D x \, d^D k}{(2\pi\hbar)^D} \, k^2 \, (\nabla\Delta)^2$$

$$N_4(E) = -\frac{1}{\pi} \frac{2\hbar^4}{D(D+2)} \, \text{Im} \int \frac{d^D x \, d^D k}{(2\pi\hbar)^D} \, k^4 \left[-(\nabla\Delta)^4 + \right.$$

$$\left. + 2 \left(\Delta(\nabla^2\Delta)\right)^2 - \frac{F^2}{\hbar^4} \Delta^4 \right]. \tag{23}$$

It should be noted that the $N_{2n}(E)$ term contains the power $\hbar^{2n-D}$. To proceed further we make use of the integrals

$$\frac{1}{\pi} \, \text{Im} \int d^D k \, k^s \, \log(k^2 + 2m(V(x) - E) + i\epsilon) =$$

$$= \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \frac{1}{s+D} \, (2m(E-V))^{\frac{D+s}{2}} \, \Theta(E-V)$$

$$\frac{1}{\pi} \, \text{Im} \int \frac{d^D k}{(2\pi)^D} \, k^{2s} \, \Delta^n = -\frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(s+\frac{D}{2}\right)}{\Gamma\left(\frac{D}{2}\right)^2 \Gamma(n)} \, (2m)^{\frac{D}{2}+s-n} \times$$

$$\times \frac{d^{n-s-1}}{dE^{n-s-1}} \left[(E-V)^{\frac{D}{2}-1} \Theta(E-V) \right], \tag{24}$$

yielding the following expression for the different contributions to the cumulative number of states

$$N_0(E) = \frac{\nu}{(4\pi)^{\frac{D}{2}} \, \Gamma\left(\frac{D}{2}+1\right)} \left(\frac{2m}{\hbar^2}\right)^{\frac{D}{2}} \int d^D x \, (E-V)^{\frac{D}{2}} \, \Theta(E-V)$$

$$N_2(E) = -\frac{\nu}{(4\pi)^{\frac{D}{2}} \, \Gamma\left(\frac{D}{2}\right)} \left(\frac{2m}{\hbar^2}\right)^{\frac{D}{2}-1} \int d^D x \, \frac{(\nabla V)^2}{12} \partial_E^2 \left[(E-V)^{\frac{D}{2}-1} \Theta(E-V) \right]$$

$$N_4(E) = \frac{\nu}{(4\pi)^{\frac{D}{2}} \, \Gamma\left(\frac{D}{2}\right)} \left(\frac{2m}{\hbar^2}\right)^{\frac{D}{2}-2} \int d^D x \, \frac{1}{60} \left\{ \frac{1}{24} (\nabla V)^4 \partial_E^5 - \frac{1}{3} (\nabla V)^2 (\nabla^2 V) \partial_E^4 + \right.$$

$$\left. + \frac{1}{2} (\nabla^2 V)^2 \partial_E^3 - \frac{5}{\hbar^4} F^2 \partial_E \right\} \left[(E-V)^{\frac{D}{2}-1} \Theta(E-V) \right]. \tag{25}$$

One can reduce the number of energy derivatives by using the identity

$$(\nabla_i V) \frac{d}{dE} f(E-V) = -\nabla_i f(E-V). \tag{26}$$

8

Integration by parts yields the result

$$
N_0(E) = \frac{\nu}{(4\pi)^{\frac{D}{2}} \Gamma\left(\frac{D}{2}+1\right)} \left(\frac{2m}{\hbar^2}\right)^{\frac{D}{2}} \int d^D x\, (E-V)^{\frac{D}{2}}\, \Theta(E-V)
$$

$$
N_2(E) = -\frac{\nu}{(4\pi)^{\frac{D}{2}} \Gamma\left(\frac{D}{2}\right)} \left(\frac{2m}{\hbar^2}\right)^{\frac{D}{2}-1} \int d^D x\, \frac{(\nabla^2 V)}{12} \partial_E \left[(E-V)^{\frac{D}{2}-1}\,\Theta(E-V)\right]
$$

$$
N_4(E) = \frac{\nu}{(4\pi)^{\frac{D}{2}} \Gamma\left(\frac{D}{2}\right)} \left(\frac{2m}{\hbar^2}\right)^{\frac{D}{2}-2} \int d^D x\, \frac{1}{288} \Big\{ (\nabla^2 V)^2 \partial_E^3 - (\nabla^4 V)\, \partial_E^2 +
$$
$$
+ \frac{2}{5} (\nabla_i \nabla_j V)^2 \partial_E^3 - \frac{24}{\hbar^4} F^2 \partial_E \Big\} \left[(E-V)^{\frac{D}{2}-1}\,\Theta(E-V)\right] .
$$

$$(27)$$

It is interesting to notice that these formulas produce finite results if the derivatives with respect to the energy are evaluated after the space integrals have been performed. If the opposite process were considered, explicit divergencies would appear due to the singularity caused by the classical turning hypersurface. After careful regularization the result would coincide with that obtained by using the previous prescription. This point can be clearly understood when dealing with a particular example (see next section).

According to the general formula (12) one can use the expressions (27) to perform a semiclassical expansion of the total energy of $N(\mu)$ fermions

$$
E_T(\mu) = E_0(\mu) + E_2(\mu) + E_4(\mu) + \ldots \tag{28}
$$

where

$$
E_0(\mu) = \mu N_0(\mu) - \frac{\nu}{(4\pi)^{\frac{D}{2}} \Gamma\left(\frac{D}{2}+2\right)} \left(\frac{2m}{\hbar^2}\right)^{\frac{D}{2}} \int d^D x\, (\mu-V)^{\frac{D}{2}+1}\, \Theta(\mu-V)
$$

$$
E_2(\mu) = \mu N_2(\mu) +
$$
$$
+ \frac{\nu}{(4\pi)^{\frac{D}{2}} \Gamma\left(\frac{D}{2}\right)} \left(\frac{2m}{\hbar^2}\right)^{\frac{D}{2}-1} \int d^D x\, \frac{(\nabla^2 V)}{12} (\mu-V)^{\frac{D}{2}-1}\, \Theta(\mu-V)
$$

$$
E_4(\mu) = \mu N_4(\mu) - \frac{\nu}{(4\pi)^{\frac{D}{2}} \Gamma\left(\frac{D}{2}\right)} \left(\frac{2m}{\hbar^2}\right)^{\frac{D}{2}-2} \int d^D x\, \frac{1}{288} \Big\{ (\nabla^2 V)^2 \partial_\mu^2 -
$$
$$
- (\nabla^4 V)\, \partial_\mu + \frac{2}{5} (\nabla_i \nabla_j V)^2 \partial_\mu^2 - \frac{24}{\hbar^4} F^2 \Big\} \left[(\mu-V)^{\frac{D}{2}-1}\,\Theta(\mu-V)\right] .
$$

$$(29)$$

Similarly to the cumulative number of states one has that $E_{2n}(\mu)$ contains a power $\hbar^{2n-D}$.

2.4 Application to the Non-Relativistic Harmonic Potential

To illustrate certain aspects of the fourth order semiclassical expansion not present in lower orders we consider the three-dimensional non-relativistic isotropic oscillator defined by the central potential

$$V(r) = \frac{1}{2} m\omega^2 r^2 \tag{30}$$

and $A_i = 0$, with eigenenergies and degeneracy given by

$$E_n = \hbar\omega \left(n + \frac{3}{2} \right), \qquad g_n = \frac{1}{2} (n+1)(n+2), \tag{31}$$

respectively. Hence the cumulative number of states is given by

$$N(E) = \nu \sum_{n=0}^{\infty} \frac{1}{2} (n+1)(n+2)\Theta(E - E_n) \tag{32}$$

where ν represents the spin degeneracy factor. The actual calculation of successive $\hbar$ corrections to $N(E)$ is simplified by direct application of formula (27). This yields

$$
\begin{aligned}
N_0(E) &= \frac{\nu}{6} \left(\frac{E}{\hbar\omega} \right)^3 \Theta(E) \\
N_2(E) &= -\frac{\nu}{8} \left(\frac{E}{\hbar\omega} \right) \Theta(E) \\
N_4(E) &= \frac{17\nu}{1920} \hbar\omega\, \delta(E)\,.
\end{aligned}
\tag{33}
$$

Notice that the result is finite but $N(E)$ has to be understood as a distribution in E. To understand how these expressions arise let us consider the following integral appearing in the fourth order contribution

$$
\begin{aligned}
N_4(E) &= \frac{17\nu}{240\pi} \hbar\omega\, \partial_E^3 \int_0^{\infty} dx\, x^2 \sqrt{E - x^2}\,\Theta(E - x^2) \\
&= \frac{17\nu}{240\pi} \hbar\omega\, \partial_E^3 \left\{ \frac{\pi}{16} E^2 \Theta(E) \right\} \\
&= \frac{17\nu}{1920} \hbar\omega\, \delta(E)\,.
\end{aligned}
\tag{34}
$$

If we had derived with respect to the energy inside the integral, then explicit divergencies would have occurred. After careful regularization the sum of the divergent terms would have produced the previous well defined result. The necessity of taking the derivative outside the integral is not directly linked to the appearance of a genuine distribution in the final formula. For instance, if only two derivatives with respect to the energy were involved in the previous calculation, the final result (proportional to $\Theta(E)$) could only be obtained by our prescription or suitable regularization. Notice that if the step function is ignored when deriving the integrand, then the final result is explicitly divergent.

Similarly one can also compute the semiclassical expansion for the total energy of a system of $N(\mu)$ fermions as defined by (12) obtaining

$$
\begin{aligned}
E_0(\mu) &= \frac{\nu}{8}\left(\frac{\mu}{\hbar\omega}\right)^3 \mu\,\Theta(\mu) \\
E_2(\mu) &= -\frac{\nu}{16}\left(\frac{\mu}{\hbar\omega}\right)\mu\,\Theta(\mu) \\
E_4(\mu) &= -\frac{17\nu}{1920}\,\hbar\omega\,\Theta(\mu)\,.
\end{aligned}
\tag{35}
$$

2.5   Reordering of the Semiclassical Expansion

From the knowledge of the number of fermions $N(\mu)$ below a chemical potential μ and the total energy $E(\mu)$ one can obtain the energy of a given number of particles A, by properly adjusting μ so that $A = N(\mu)$. In what follows we will discuss a procedure within the semiclassical approach, which is advantageous for practical calculations [26, 25]. This method can be applied both in the non-relativistic and the relativistic cases. Our starting point is the former semiclassical expansion for the number of particles and the total energy

$$
\begin{aligned}
N(\mu) &= N_0(\mu) + N_2(\mu) + N_4(\mu) + \dots \\
E(\mu) &= E_0(\mu) + E_2(\mu) + E_4(\mu) + \dots \, .
\end{aligned}
\tag{36}
$$

From the normalization condition $A = N(\mu)$ it is clear that μ inherits a certain $\hbar$ dependence. We proceed along the lines proposed in previous works [26, 25], by further expanding the chemical potential into powers of $\hbar$

$$
\mu = \mu_0 + \mu_2 + \mu_4 + \dots \, ,
\tag{37}
$$

with the lowest order condition

$$
N_0(\mu_0) = A \, .
\tag{38}
$$

It should be clear that by doing so we are simply reordering the semiclassical series. If the expansion is truncated at a certain finite order in $\hbar$, the result of this reordering is consistent up to higher $\hbar$ corrections. One should also mention that for this expansion to be a true $\hbar$-expansion, i.e. that μ_0 is independent of $\hbar$, one has to declare that A is of order $\hbar^{-D}$ since this is the power which appears in N_0. We will always assume this particular order counting. If we insert equations (37) and (38) back into the first line of equation (36) the successive terms in the expansion of μ are automatically fixed giving

$$
\begin{aligned}
\mu_0 \quad &: \quad N_0(\mu_0) = A \\
\mu_2 \quad &\equiv \quad -\frac{N_2(\mu_0)}{N_0'(\mu_0)} \\
\mu_4 \quad &\equiv \quad -\frac{N_4(\mu_0)}{N_0'(\mu_0)} + \frac{N_2(\mu_0)\,N_2'(\mu_0)}{[N_0'(\mu_0)]^2} - \frac{N_0''(\mu_0)\,[N_2(\mu_0)]^2}{2\,[N_0'(\mu_0)]^3}\,.
\end{aligned}
\tag{39}
$$

Similarly one can expand the total energy to give

$$
E_T(\mu) \;=\; \mathcal{E}_0(\mu_0) + \mathcal{E}_2(\mu_0) + \mathcal{E}_4(\mu_0) + \mathcal{O}(\hbar^{6-D})\,,
\tag{40}
$$

where

$$
\begin{aligned}
\mathcal{E}_0(\mu_0) &\equiv E_0(\mu_0) \\
\mathcal{E}_2(\mu_0) &\equiv E_2(\mu_0) + \mu_2(\mu_0)\,E_0'(\mu_0) \\
\mathcal{E}_4(\mu_0) &\equiv E_4(\mu_0) + \mu_4(\mu_0)\,E_0'(\mu_0) + \frac{1}{2}\,[\mu_2(\mu_0)]^2\,E_0''(\mu_0) + \mu_2(\mu_0)\,E_2'(\mu_0)\,.
\end{aligned}
\tag{41}
$$

If we now apply the relationship between the number of particles and the total energy as given by eq. (12) one obtains after deriving with respect to μ

$$
\begin{aligned}
E_m'(\mu) &= \mu\,N_m'(\mu) \\
E_m''(\mu) &= N_m'(\mu) + \mu\,N_m''(\mu)\,.
\end{aligned}
\tag{42}
$$

Using this result we get the final simple expressions

$$
\begin{aligned}
\mathcal{E}_0(\mu_0) &\equiv E_0(\mu_0) \\
\mathcal{E}_2(\mu_0) &\equiv E_2(\mu_0) - \mu_0\,N_2(\mu_0) \\
\mathcal{E}_4(\mu_0) &\equiv E_4(\mu_0) - \mu_0\,N_4(\mu_0) + \frac{1}{2}\,\frac{[N_2(\mu_0)]^2}{N_0'(\mu_0)}\,.
\end{aligned}
\tag{43}
$$

The advantages of this reordering in the total energy are manifold. First, the chemical potential has to be adjusted only once, namely at lowest and simplest order. Second, there is a clear order by order separation, that allows to treat the corrections in the energy additively. Finally, big cancellations are avoided in practical calculations since the cumulative number turns out to be extremely sensitive to variations in the chemical potential.

3   Relativistic Case

3.1   *Cumulative Number for the Dirac equation*

In this section we will consider the semiclassical expansion of the Dirac Hamiltonian with external vector and scalar fields. The results derived here will be applied to estimate higher WK$\hbar$ corrections in a mean field relativistic nuclear model.

The Dirac eigenvalue problem of a particle in D dimensions can be written as

$$
\begin{aligned}
H\,\psi_{n\alpha} &= E_n\,\psi_{n\alpha} \\
H &\equiv \alpha\cdot P + \beta\,\Phi(x) + V(x)\,,
\end{aligned}
\tag{44}
$$

where $\alpha\cdot P \equiv \sum_{i=1}^{D}\alpha^i P^i$ with $\alpha^i \equiv \gamma^0\gamma^i$ and $\beta = \gamma^0$. The gamma matrices satisfy the usual anticommutation relation $\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}$ where the Minkowskian metric is $(1,-1,\ldots,-1)$ and the generalized momentum P^i is given by

$$
P^i \equiv -i\,\hbar\,\partial_i - A^i(x)\,.
\tag{45}
$$

The time independent potentials $\Phi(x)$ and $V(x)$ represent the scalar field and the time component of a (D+1)-vector field respectively. The rest mass of the particle m is included in the scalar field through the asymptotic condition $\Phi(\infty) = m$. The cumulative number of states is defined as

$$
N^H(E) = \sum_n g_n\,\Theta(E - E_n) = \mathrm{Tr}\,\Theta(E - H),
\tag{46}
$$

where g_n represents the possible degeneracy of the eigenvalue E_n. The Dirac Hamiltonian is not bounded from below and hence the cumulative number becomes infinite. Nevertheless one can redefine this quantity by suitably choosing the energy for which the number of states is counted from. This operation corresponds to a certain normal ordering. In the free case, the Dirac equation exhibits an energy gap of width $2m$, which distinguishes between positive and negative energy eigenvalues. In the interacting case it seems reasonable to assume that, if the fields V and Φ are not too strong compared to the rest mass m, some gap in the Dirac spectrum exists.

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For the potentials which usually appear in relativistic nuclear physics this turns out to be the case. We will see that in the semiclassical limit there might appear an energy gap if the interval $[\max\{V(x) - \Phi(x)\}, \min\{V(x) + \Phi(x)\}]$ is not empty. The former discussion suggests the subtraction point to lie within the gap

$$N(E) \equiv N^H(E) - N^H(E_{\text{gap}}). \tag{47}$$

Furthermore, it is convenient to split $N(E)$ into the regions above (Ω_+) and below (Ω_-) the gap as follows

$$
\begin{aligned}
N(E) &= N_+(E) - N_-(E) \\
N_+(E) &\equiv \sum_{n \in \Omega_+} g_n \, \Theta(E - E_n) \\
N_-(E) &\equiv \sum_{n \in \Omega_-} g_n \, \Theta(E_n - E).
\end{aligned}
\tag{48}
$$

It should be noticed that with this definition both $N_+(E)$ and $N_-(E)$ are non-negative numbers. The level density is then given by

$$
\begin{aligned}
g(E) &\equiv \frac{dN(E)}{dE} = \frac{dN^H(E)}{dE} \\
&= \sum_{n \in \Omega} g_n \, \delta(E - E_n),
\end{aligned}
\tag{49}
$$

where the irrelevant constant $N^H(E_{\text{gap}})$ disappears. To evaluate the semiclassical expansion of $N(E)$ we will deal directly with $N^H(E)$ and then we will proceed to perform the subtraction and the separation between the two branches (above and below the gap).

3.2   *Independent Particle Approximation*

Notice that all manipulations above are intrinsic to the Dirac spectrum and do not yet define a many body picture. In the spirit of the Hartree approximation (independent particle model) we introduce a chemical potential as we did in the non-relativistic case. However, the total energy computed as the sum of all eigenstates below the chemical potential is an ultraviolet divergent quantity, which cannot be made finite by simply subtracting the vacuum contribution from the Dirac sea. In a renormalizable field theory, one could subtract additional counterterms to the total energy. To keep the line of reasoning straight we relegate the discussion of the Dirac sea corrections to section 3.5. For most of this paper we will work in the Hartree valence approximation, where many calculations have been done. As we will show

the semiclassical Dirac sea corrections can be formally obtained from the valence contribution. From the point of view of the many body theory this approximation corresponds to deal with a non-relativistic many body problem using relativistic kinematics.

The valence energy of a system of A-fermions with energies between the chemical potential and the energy gap is given by

$$E_T(\mu) = \mu\, N(\mu) - \int dE\, N(E)\, \Theta(\mu - E) \tag{50}$$

with the normalization condition

$$A = N(\mu). \tag{51}$$

In evaluating several single particle densities, it is necessary to make use of a generalization of the eq. (14) to the relativistic case

$$\frac{\delta N_\pm(E)}{\delta V(x)} = \mp \sum_{n \in \Omega_\pm} \sum_{\alpha=1}^{g_n} \psi_{n\alpha}^\dagger(x)\, \psi_{n\alpha}(x)\, \delta(E - E_n) \tag{52}$$

$$\frac{\delta N_\pm(E)}{\delta \Phi(x)} = \mp \sum_{n \in \Omega_\pm} \sum_{\alpha=1}^{g_n} \psi_{n\alpha}^\dagger(x)\, \beta\, \psi_{n\alpha}(x)\, \delta(E - E_n). \tag{53}$$

Thus, the valence energy density is given by

$$
\begin{aligned}
\rho_E(x, \mu) &= \rho_E^+(x, \mu) - \rho_E^-(x, \mu) \\
\rho_E^\pm(x, \mu) &= \sum_{n \in \Omega_\pm} \sum_{\alpha=1}^{g_n} \psi_{n\alpha}^\dagger(x)\, E_n\, \psi_{n\alpha}(x)\, \Theta\left(\pm(\mu - E_n)\right) \\
&= \mp \int dE\, \frac{\delta N_\pm(E)}{\delta V(x)}\, E\, \Theta\left(\pm(\mu - E)\right),
\end{aligned}
\tag{54}
$$

the valence particle density is

$$
\begin{aligned}
\rho(x, \mu) &= \rho^+(x, \mu) - \rho^-(x, \mu) \\
\rho^\pm(x, \mu) &= \sum_{n \in \Omega_\pm} \sum_{\alpha=1}^{g_n} \psi_{n\alpha}^\dagger(x)\, \psi_{n\alpha}(x)\, \Theta\left(\pm(\mu - E_n)\right) \\
&= \mp \int dE\, \frac{\delta N_\pm(E)}{\delta V(x)}\, \Theta\left(\pm(\mu - E)\right),
\end{aligned}
\tag{55}
$$

and the valence scalar density reads

$$\rho_s(x, \mu) = \rho_s^+(x, \mu) - \rho_s^-(x, \mu)$$

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$$\rho_s^{\pm}(x,\mu) \;\; = \;\; \sum_{n \in \Omega_{\pm}} \sum_{\alpha=1}^{g_n} \psi_{n\alpha}^{\dagger}(x)\, \beta \, \psi_{n\alpha}(x)\, \Theta\left(\pm\left(\mu - E_n\right)\right)$$

$$= \;\; \mp \int dE \, \frac{\delta N_{\pm}(E)}{\delta \Phi(x)}\, \Theta\left(\pm\left(\mu - E\right)\right). \tag{56}$$

The kinetic energy density could be readily obtained from the former ones by means of the formula

$$\rho_K^{+}(x,\mu) \;\; = \;\; \sum_{n \in \Omega_{+}} \sum_{\alpha=1}^{g_n} \psi_{n\alpha}^{\dagger}(x)\, (\alpha \cdot P + \beta\, m - m)\, \psi_{n\alpha}(x)\, \Theta\left(\mu - E_n\right)$$

$$= \;\; \rho_E^{+}(x,\mu) + (m - \Phi(x))\, \rho_s^{+}(x,\mu) - (m + V(x))\, \rho^{+}(x,\mu). \tag{57}$$

3.3 Semiclassical Expansion

Following previous work [12] we can apply a semiclassical expansion to the relativistic case. A direct application of the logarithmic regularization to the step function gives

$$N^H(E) \;\; = \;\; \frac{1}{\pi}\, \mathrm{Im}\, \mathrm{Tr} \log(H - E + i\,\epsilon)$$

$$= \;\; \frac{1}{\pi}\, \arg \det(H - E + i\,\epsilon). \tag{58}$$

which does not allow yet to employ the general method described in the appendix. To do so, we consider the transformation of the Dirac operator in an odd number of dimensions

$$\tilde{H} \;\; \equiv \;\; (\beta\gamma_5)\, H\, (\beta\gamma_5)^{-1}$$

$$= \;\; -\alpha \cdot P - \beta\, \Phi(x) + V(x)\,, \tag{59}$$

where $\gamma_5 \equiv -i\, \gamma_0\gamma_1 \cdots \gamma_D = \gamma_5^{-1}$. Using the properties of the determinant one obtains that

$$\arg \det(H - E + i\,\epsilon) = \frac{1}{2}\, \arg \det\left[(H - E + i\,\epsilon)\,(\tilde{H} - E + i\,\epsilon)\right], \tag{60}$$

whence

$$N^H(E) = \frac{1}{2\pi}\, \mathrm{Im}\, \mathrm{Tr} \log(-P^2 - U(X))\,, \tag{61}$$

where the following definitions have been introduced

$$U(X) \;\; \equiv \;\; \Phi^2 - (E - V - i\,\epsilon)^2 + i\,\hbar\, \alpha \cdot \nabla(V - \beta\,\Phi) - \frac{1}{2}\sigma_{ij}\, F_{ij}$$

$$\sigma_{ij} \;\; \equiv \;\; \frac{i}{2}\left[\gamma^i, \gamma^j\right]$$

$$F_{ij} \;\; \equiv \;\; -i\left[P^i, P^j\right] = \hbar\left(\nabla_i A^j - \nabla_j A^i\right). \tag{62}$$

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The former equation is now ready for a semiclassical treatment, since it involves an elliptic operator with some spinorial structure. Up to fourth order in the WK expansion a direct application of table 9 yields

$$
\begin{aligned}
N^H(E) &= \frac{1}{2\pi}\,\mathrm{Im}\int \frac{d^D x\,d^D k}{(2\pi\hbar)^D}\,\mathrm{tr}\left\{\log(-k^2 - U) + \frac{k^2\,\hbar^2}{D}\,(\nabla\Delta)^2 -\right.\\
&\quad - \frac{2\,k^4\,\hbar^4}{D(D+2)}\left[2\left(\Delta\left(\nabla^2\Delta\right)\right)^2 + ((\nabla_i\Delta)\,(\nabla_j\Delta))^2 - 2\,(\nabla\Delta)^4 -\right.\\
&\quad \left.\left. - \frac{1}{\hbar^4}\left(F_{ij}\Delta^2\right)^2 - \frac{4}{\hbar^2}\,iF_{ij}\,\Delta(\nabla_i\Delta)\,(\nabla_j\Delta)\,\Delta\right] + \mathcal{O}(P^6)\right\},
\end{aligned}
$$

(63)

where the matrix propagator $\Delta(x,k)^{-1} \equiv k^2 + U(x)$ has been defined. This operator contains some $\hbar$ dependence, so that we reexpand around

$$
\Delta_o(x,k)^{-1} = k^2 + \Phi^2 - (E - V - i\epsilon)^2.
$$

(64)

Direct calculation of the Dirac traces together with integration by parts yields

$$
N^H(E) = N_0^H(E) + N_2^H(E) + N_4^H(E) + \Theta(\hbar^{6-D})
$$

(65)

where

$$
\begin{aligned}
N_0^H(E) &= \frac{T(D)}{2\pi}\,\mathrm{Im}\int \frac{d^D x\,d^D k}{(2\pi\hbar)^D}\,\log(-\Delta_o^{-1})\\
N_2^H(E) &= \frac{T(D)\hbar^2}{2\pi}\,\mathrm{Im}\int \frac{d^D x\,d^D k}{(2\pi\hbar)^D}\left\{\frac{1}{2}\left[(\nabla V)^2 - (\nabla\Phi)^2\right]\Delta_o^2 +\right.\\
&\quad \left. + \frac{k^2}{D}\,(\nabla\Delta_o)^2\right\}\\
N_4^H(E) &= \frac{T(D)\hbar^4}{2\pi}\,\mathrm{Im}\int \frac{d^D x\,d^D k}{(2\pi\hbar)^D}\left\{-\frac{1}{4}\Delta_o^4\left[(\nabla V)^4 + (\nabla\Phi)^4 -\right.\right.\\
&\quad \left. - 6\,(\nabla V)^2\,(\nabla\Phi)^2 + 4(\nabla V\cdot\nabla\Phi)^2\right] +\\
&\quad + \frac{k^2}{D}\left[-\left(\nabla_i\left(\Delta_o^2\nabla_j V\right)\right)^2 + \left(\nabla_i\left(\Delta_o^2\nabla_j\Phi\right)\right)^2 +\right.\\
&\quad \left. + 2\,\Delta_o^3\,(\nabla^2\Delta_o)\left((\nabla V)^2 - (\nabla\Phi)^2\right)\right] -\\
&\quad \left. - \frac{2k^4}{D(D+2)}\left[2\,\Delta_o^2\,(\nabla^2\Delta_o)^2 - (\nabla\Delta_o)^4 + \frac{2}{\hbar^4}\,F^2\,\Delta_o^4\right]\right\}.
\end{aligned}
$$

(66)

Here $T(D)$ is the spinor space dimension ($T(1) = 2$; $T(3) = 4$). To work out the momentum integrals we note that any power of Δ_o can be obtained as

$$\Delta_o^n = \frac{(-1)^{n-1}}{\Gamma(n)} \, \partial_{\Phi^2}^n \log(-\Delta_o^{-1}). \tag{67}$$

We also use the identity

$$\lim_{\epsilon \to 0^+} \frac{1}{\pi} \operatorname{Im} \log(x - i\,y\,\epsilon) = 1 + \operatorname{sign}(y) \, \Theta(x), \tag{68}$$

and define the local momentum

$$p^2(x) \equiv (E - V(x))^2 - \Phi(x)^2. \tag{69}$$

It is interesting to notice that within a semiclassical treatment one can always define a gap if the interval $[\max\{V(x) - \Phi(x)\}, \min\{V(x) + \Phi(x)\}]$ is not empty. In practice this allows to explicitly distinguish the contribution of states below and above the gap in the semiclassical limit by means of the formula

$$
\begin{aligned}
1 + \operatorname{sign}(E - V)\,\Theta(p^2 - k^2) &= \\
&= 1 + \Theta(E - V - \sqrt{\Phi^2 + k^2}) - \Theta\left(-(E-V) - \sqrt{\Phi^2 + k^2}\right).
\end{aligned}
$$

The momentum integrals which appear in the semiclassical expansion have the form

$$
\begin{aligned}
I_D^{n,s}(E) &= \frac{1}{\pi} \operatorname{Im} \int \frac{d^D k}{(2\pi)^D} \, k^{2s} \Delta_o^n \\
&= \frac{(-1)^{n-1}\,2}{(4\pi)^{D/2}\,\Gamma(n)\Gamma\left(\frac{D}{2}\right)} \, \partial_{\Phi^2}^n \int_0^\infty dk \, k^{2s+D-1} \, \times \\
&\quad \times \left[1 + \Theta(E - V - \sqrt{\Phi^2 + k^2}) - \Theta\left(-(E-V) - \sqrt{\Phi^2 + k^2}\right)\right].
\end{aligned}
$$

If we further subtract the contribution from the gap and explicitly separate the two branches we get for $n > 0$

$$I_D^{n,s}(E) - I_D^{n,s}(E_{\mathrm{gap}}) = I_D^{+,n,s}(E) - I_D^{-,n,s}(E) \tag{70}$$

where

$$I_D^{\pm,n,s}(E) = \frac{(-1)^{n-s}\,\Gamma\left(s + \frac{D}{2}\right)}{(4\pi)^{D/2}\,\Gamma(n)\,\Gamma\left(\frac{D}{2}\right)^2} \, \partial_{\Phi^2}^{n-s-1} \left[p^{D-2}\,\Theta(\pm(E-V) - \Phi)\right]. \tag{71}$$

The zeroth order term can be evaluated following similar steps. After suitable integration by parts, the final result for the cumulative number of states reads

$$N_0^\pm(E) = \frac{T(D)}{2} \frac{1}{(4\pi\hbar^2)^{D/2}\Gamma(\frac{D}{2}+1)} \int d^D x \, p^D \, \Theta(\pm(E-V)-\Phi)$$

$$N_2^\pm(E) = -\frac{T(D)}{2} \frac{\hbar^2}{(4\pi\hbar^2)^{D/2}\Gamma(D/2)} \int d^D x \left\{ \frac{1}{2}[(\nabla\Phi)^2 - (\nabla V)^2]\partial_{\Phi^2} + \right.$$

$$\left. + \frac{1}{3}[\Phi(\nabla\Phi) + (E-V)(\nabla V)]^2 \partial_{\Phi^2}^2 \right\} p^{D-2}\Theta(\pm(E-V)-\Phi)$$

$$N_4^\pm(E) = -\frac{T(D)}{2} \frac{\hbar^4}{(4\pi\hbar^2)^{D/2}\Gamma(D/2)} \int d^D x \left\{ \vphantom{\frac{1}{90}} \right.$$

$$\frac{1}{90}[\Phi(\nabla\Phi) + (E-V)(\nabla V)]^4 \partial_{\Phi^2}^5 + \frac{1}{12}[(\nabla^2\Phi)^2 - (\nabla^2 V)^2]\partial_{\Phi^2}^2 +$$

$$+ \frac{1}{24}\left[(\nabla V)^4 + (\nabla\Phi)^4 - 6(\nabla V)^2(\nabla\Phi)^2 + 4(\nabla V \cdot \nabla\Phi)^2\right]\partial_{\Phi^2}^3 +$$

$$+ \frac{1}{12}[(\nabla V)^2 - (\nabla\Phi)^2][(\nabla\Phi)^2 + \Phi(\nabla^2\Phi) - (\nabla V)^2 + (E-V)(\nabla^2 V)]\partial_{\Phi^2}^3 +$$

$$+ \frac{1}{6}\left[(\nabla^2\Phi)(\nabla_i\Phi) - (\nabla^2 V)(\nabla_i V)\right][\Phi(\nabla_i\Phi) + (E-V)(\nabla_i V)]\partial_{\Phi^2}^3 +$$

$$+ \frac{2}{45}[\Phi(\nabla\Phi) + (E-V)(\nabla V)]^2 \times$$

$$\times \left[(\nabla\Phi)^2 + \Phi(\nabla^2\Phi) - (\nabla V)^2 + (E-V)(\nabla^2 V)\right]\partial_{\Phi^2}^4 +$$

$$+ \frac{1}{30}\left[(\nabla\Phi)^2 + \Phi(\nabla^2\Phi) - (\nabla V)^2 + (E-V)(\nabla^2 V)\right]^2 \partial_{\Phi^2}^3 +$$

$$+ \frac{1}{6}\frac{F^2}{\hbar^4}\partial_{\Phi^2} \right\} p^{D-2}\Theta(\pm(E-V)-\Phi) , \tag{72}$$

which has been already obtained in a previous work [12]. In practical calculations it is desirable to lower the number of derivatives acting on the step function. This can be achieved by using the identity

$$\nabla_i \left[f(p)\,\Theta(\pm(E-V)-\Phi)\right] =$$
$$= 2\left[\Phi(\nabla_i\Phi) + (E-V)(\nabla_i V)\right]\partial_{\Phi^2}\left[f(p)\,\Theta(\pm(E-V)-\Phi)\right], \tag{73}$$

together with integration by parts yielding

$$N_0^\pm(E) = \frac{T(D)}{2} \frac{1}{(4\pi\hbar^2)^{D/2}\Gamma(\frac{D}{2}+1)} \int d^D x \, p^D \Theta(\pm(E-V)-\Phi)$$

$$
\begin{aligned}
N_2^\pm(E) &= \frac{T(D)}{2}\frac{\hbar^2}{(4\pi\hbar^2)^{D/2}\Gamma(D/2)}\int d^D x \left\{\frac{1}{2}[(\nabla V)^2 - (\nabla\Phi)^2]-\right.\\
&\quad\left. -\frac{1}{12}\nabla^2 p^2\right\}\partial_{\Phi^2} p^{D-2}\Theta(\pm(E-V)-\Phi)\\
N_4^\pm(E) &= \frac{T(D)}{2}\frac{\hbar^4}{(4\pi\hbar^2)^{D/2}\Gamma(D/2)}\int d^D x \left\{\right.\\
&\quad -\frac{1}{288}\left[(\nabla^2 p^2) + 12\,(\nabla\Phi)^2 - 12\,(\nabla V)^2\right](\nabla^2 p^2)\,\partial_{\Phi^2}^3 - \frac{1}{720}\,(\nabla_i\nabla_j p^2)^2\partial_{\Phi^2}^3 -\\
&\quad -\frac{1}{24}\left[(\nabla V)^4 + (\nabla\Phi)^4 - 6\,(\nabla V)^2\,(\nabla\Phi)^2 + 4\,(\nabla V\cdot\nabla\Phi)^2\right]\partial_{\Phi^2}^3 +\\
&\quad +\frac{1}{288}\,(\nabla^4 p^2)\,\partial_{\Phi^2}^2 + \frac{1}{12}[(\nabla_i\Phi)\,(\nabla_i\nabla^2\Phi) - (\nabla_i V)\,(\nabla_i\nabla^2 V)]\partial_{\Phi^2}^2 -\\
&\quad \left. -\frac{1}{6}\frac{F^2}{\hbar^4}\partial_{\Phi^2}\right\}p^{D-2}\Theta(\pm(E-V)-\Phi)
\end{aligned}
$$

$$(74)$$

Again, the final result for $N(E)$ is explicitly convergent if the derivatives with respect to the scalar field are evaluated after space integration, as can be seen by introducing a new parameter λ

$$
\begin{aligned}
&\int d^D x\, g[x,\Phi]\,\partial_{\Phi^2}^n\left[f[\Phi^2]\,\Theta(\pm(E-V)-\Phi)\right] =\\
&= \partial_{\lambda^2}^n \int d^D x\, g[x,\Phi]\, f[\Phi^2 + \lambda^2]\,\Theta(\pm(E-V)-\sqrt{\Phi^2+\lambda^2})\Big|_{\lambda=0}.
\end{aligned}
\qquad (75)
$$

It is interesting to notice that the non-relativistic limit of the former expressions yields that one given in section (2.3). The only difference stems from the spin interaction with the gauge field. For instance for $D = 3$ the correct non-relativistic limit of the Dirac equation is not the Schrödinger equation but the Pauli equation which contains a dipole spin interaction $-\frac{\hbar}{2m}\,\sigma\cdot(\nabla\wedge\mathbf{A})$. This has its corresponding effect in the magnetic polarizability terms, since the pure non-relativistic treatment gives

$$
N_4^{\mathrm{mag}}(E) = -\frac{1}{48\,\pi^2\,\hbar^3\,\sqrt{2m}}\int d^3 x\,\frac{F^2}{(E-V)^{1/2}}\,\Theta(E-V)\,,
\qquad (76)
$$

and the non-relativistic limit of the Dirac equation produces

$$
N_4^{\mathrm{mag}}(E) = \frac{1}{24\,\pi^2\,\hbar^3\,\sqrt{2m}}\int d^3 x\,\frac{F^2}{(E-V)^{1/2}}\,\Theta(E-V)\,.
\qquad (77)
$$

The explicit expressions for the upper branch contribution to the various densities are rather lengthy and are given in tables 1, 2, and 3, where the definitions

$$
\epsilon_F(x,\mu) \equiv \mu - V(x)
$$

$$
\begin{aligned}
k_F(x,\mu) &\equiv \sqrt{\epsilon_F^2 - \Phi(x)^2} \\
x_F(x,\mu) &\equiv \frac{\epsilon_F}{k_F} \\
l_F(x,\mu) &\equiv \log \frac{\epsilon_F + k_F}{\Phi(x)} \,.
\end{aligned}
\tag{78}
$$

have been used. A few remarks are in order. The expressions have been written in the simplest possible way and following the standard fashion. However, they appear to be explicitly divergent at the classical turning hypersurface and they would produce non convergent mean values. In fact, when writing those expressions, some purely distributive terms have been intentionally omitted. They can be recovered by means of the rules

$$
\begin{aligned}
x_F &\rightarrow \frac{\mu - V}{k_F} \\
k_F^{-n}\,\Theta(\mu - V - \Phi) &\rightarrow -2^n \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n)} \partial_{\Phi^2}^{\frac{n+1}{2}}\left[k_F\,\Theta(\mu - V - \Phi)\right]
\end{aligned}
\tag{79}
$$

(notice that n is always an odd number so that the derivative is evaluated an integer number of times). It can be checked that then any negative power of k_F produces finite results for the mean values of the densities although the density itself still remains divergent at the classical turning hypersurface. Similarly to the cumulative number it is convenient to replace the scalar field derivatives by introducing an external parameter λ so that the space integration can be performed first. Obviously such a procedure becomes rather involved in practical numerical applications. In a forthcoming section we will present a trick specially suited for numerical calculations and valid in the case of central potentials, that avoids the use of the substitution rules given above, hence allowing for a literal usage of tables 1-3.

Finally, let us remark that the case $\Phi = m = \text{constant}$, $A = 0$, corresponds to a relativistic atomic system. For such a system the semiclassical expansion up to $\hbar^4$ order has been performed [16] in the spirit of density functional theory. For this particular case our formulas in table 1 and 3 coincide (up to purely distributional terms from eq. (79)) with their intermediate WK results.

3.4   Application to Relativistic Harmonic Potentials

In previous work [5, 11] harmonic potentials in the Dirac equation have been used to illustrate the WK expansion up to second order in $\hbar$ in the total energy, and a comparison with the exact results as well as with the Strutinsky average [30, 20] have been carried out. At this order of $\hbar$ many of the problems discussed above do

not take place, in particular the distributional character of the cumulative number. In this section we proceed to apply our formalism to the harmonic potentials up to fourth order in $\hbar$ and for $D = 3$ space dimensions along the lines of [12]. More specifically we consider the central potentials

$$V(r) = \Phi(r) - m = \frac{1}{4} m\omega^2 r^2\,, \qquad A_i(r) = 0\,. \tag{80}$$

The Dirac eigenvalue problem can be solved for the bound states of positive energy. The energy of the n-th excited state reads

$$(E_n^+ - m)(E_n^{+2} - m^2) = 2m\omega^2 \left(n + \frac{3}{2}\right)^2\,, \tag{81}$$

and has degeneracy $g_n = (n+1)(n+2)$, hence the cumulative number of states reads

$$N^+(E) = \sum_{n=0}^{\infty} (n+1)(n+2)\,\Theta(E - E_n^+)\,. \tag{82}$$

Straightforward application of our formulas yields

$$
\begin{aligned}
N_0^+(\mu) &= \frac{8}{3} m^3 \left(\frac{m}{\hbar\omega}\right)^3 (1+x)^{\frac{3}{2}} x^3\, \Theta(\mu - m) \\
N_2^+(\mu) &= -\frac{1}{2} m \left(\frac{m}{\hbar\omega}\right) (1+x)^{\frac{1}{2}} x\, \Theta(\mu - m) \\
N_4^+(\mu) &= \frac{17}{960} \hbar\omega\, \delta(\mu - m)\,,
\end{aligned}
\tag{83}
$$

where the scaled dimensionless variable $x = (\mu - m)/(2m)$ has been introduced. Note that the fourth order term is a pure distribution. Similarly the total energy up to fourth order reads

$$
\begin{aligned}
E_0^+(\mu) &= \frac{512}{3465} m \left(\frac{m}{\hbar\omega}\right)^3 (1+x)^{\frac{1}{2}} \left[\frac{945}{32} x^5 + \frac{2905}{64} x^4 + \right. \\
&\quad \left. + \frac{1135}{64} x^3 + \frac{3}{8} x^2 - \frac{1}{2} x + 1 - (1+x)^{-\frac{1}{2}}\right] \Theta(\mu - m) \\
E_2^+(\mu) &= -\frac{4}{15} m \left(\frac{m}{\hbar\omega}\right) (1+x)^{\frac{1}{2}} \left[\frac{9}{4} x^2 + \frac{11}{8} x + 1 - \right. \\
&\quad \left. - (1+x)^{-\frac{1}{2}}\right] \Theta(\mu - m) \\
E_4^+(\mu) &= -\frac{17}{960} \hbar\omega\, [\Theta(\mu - m) - m\,\delta(\mu - m)]\,.
\end{aligned}
\tag{84}
$$

It is interesting to note that the distributive contribution in the fourth order term corresponds to the fourth order correction of $mN(\mu)$ which is the total energy of $N(\mu)$ particles and hence cancels in the expression for the binding energy. We ignore whether this is a general feature of the semiclassical expansion. As a side remark let us mention that the non-relativistic reduction of the former expressions coincides with the non-relativistic result of section 2.4, corresponding to the spin degeneracy $\nu = 2$ and a non-relativistic potential $V+\Phi-m$, provided the rest mass contributions for the energy and the chemical potential are subtracted.

The semiclassical expansion for this particular example can be checked directly (as suggested in [26, 20]) by means of the asymptotic Euler-Maclaurin summation formula [31]

$$\sum_{n=0}^{\infty} F(n) = \int_0^{\infty} dt\, F(t) + \frac{1}{2}\left[F(0) + F(\infty)\right] + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}\left[F^{(2n-1)}(\infty) - F^{(2n-1)}(0)\right],$$

(85)

where B_m stand for the Bernoulli numbers.

To have a quantitative idea of how large are the corrections we take $m = 939$ MeV and fix the oscillator constant through the relation $\hbar\omega = 41A^{-\frac{1}{3}}$ MeV. We also consider two nucleon species so that $A = N + Z$ and $N = Z$. Our results are presented in table 4. We consider two versions of the semiclassical expansion and compare them to the exact result and the Strutinsky average. In any case we consider the total number of particles to be $A = N(\mu)$. In the first version of the semiclassical expansion, denoted as non-perturbative (NP), we proceed as follows. For a given order in $\hbar$, say $\hbar^{2k}$ we fix numerically the chemical potential μ directly from the normalization equation $A = N_0(\mu)+\ldots+N_{2k}(\mu)$ and recompute the energy up to that order as $E(\mu) = E_0(\mu)+\ldots+E_{2k}(\mu)$. This is done independently for each order. In the second method, denoted as perturbative (P), we proceed along the lines described in section (2.5) [26, 25]. We fix the zeroth order condition $A = N_0(\mu_0)$ and compute the corrections in $\hbar$ to it by expanding $\mu = \mu_0 + \mu_2 + \ldots$. This second procedure has the practical advantage of making the contributions of different orders to be additive. We also construct the Strutinsky averaged (SA) energy since it is commonly believed that the semiclassical expansion should converge to it, and not to the exact energy since shell effects have been smoothed out. To compute this average we have chosen a Gaussian weight function. As can be seen from the table the semiclassical expansion seems indeed to converge to the Strutinsky average in this particular example at least up to order WK$\hbar^4$. We also observe a clear trend between the non-perturbative and the perturbative determination of the chemical potential, namely there appears not to be a substantial improvement in the numerical value of the energy. However, one has to say that the perturbative method avoids big cancellations as it turns out to be the case in the non-perturbative procedure, and

hence it is better suited for numerical estimates. Incidentally, let us mention that our non-perturbative calculation up to WK$\hbar^2$ coincides numerically with previous estimates [11] (we only find a small deviation from our Strutinsky average[4] to theirs in the case $A = 16$). Finally, let us point out that the fourth order contribution is much smaller (~ 0.5 MeV) than the difference between the semiclassical expansion and the exact result ($\sim 10 - 20$ MeV). This is however not the case for the second order in $\hbar$ which turns out to be even four times larger than the shell effects.

3.5   *Semiclassical Dirac Sea Corrections*

As already mentioned, the Dirac sea contribution to the energy, defined as the sum of all Dirac eigenvalues below zero energy, is an ultraviolet divergent quantity which cannot be made finite by simple subtraction of the vacuum energy. However, the finite contributions of the sea energy acquire a precise meaning within a given renormalization scheme in a renormalizable quantum field theory, the infinite parts corresponding to absorb the infinities in the parameters of the Lagrangian. This is the case for instance in the σ-ω model. Nevertheless, it has been shown that for relativistic Lagrangians of fermions interacting through Yukawa like couplings, the renormalization program cannot be carried out consistently within the approximation scheme developed so far. Indeed, a short distance vacuum instability of Landau type takes place at the zero boson loop one fermion loop level [32, 33], so that tachyonic poles appear. It is not clear whether this is a true feature of this kind of Lagrangians or simply an artifact of the approximate method used to derive it. This seems to suggest that it is more sensible to stick within a valence approximation. On the other hand, such an approximation breaks unitarity and Lorentz covariance (see the discussion in [4] and references therein). Since we are dealing with an effective theory it is not clear what is better, to neglect the Dirac sea or to face vacuum instabilities. In this section we consider for future reference the WK$\hbar^4$-corrections to the Dirac sea densities.

From the point of view of the semiclassical expansion, such a short distance instability can never be seen, due to the large distance nature of the semiclassical approach. In addition, the ultraviolet divergencies correspond to well defined orders in $\hbar$, namely the orders WK$\hbar^0$ and WK$\hbar^2$ present quadratic and logarithmic divergencies respectively, and higher orders turn out to be convergent. In what follows we will derive the WK$\hbar^4$ corrections to the sea energy from the explicit expressions obtained within the valence approximation. First, we note that charge conjugation

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[4]We stress the fact that the number of significant digits presented in our table are compatible with the so called plateau condition [20].

symmetry implies

$$N_-(E, V) = N_+(-E, -V) \tag{86}$$

Using this equation and (54-56) the relations for the baryon, scalar and total energy densities are

$$
\begin{align*}
\rho^+(x, \mu, V) &= \rho^-(x, -\mu, -V) \\
\rho_s^+(x, \mu, V) &= -\rho_s^-(x, -\mu, -V) \\
\rho_E^+(x, \mu, V) &= -\rho_E^-(x, -\mu, -V) \tag{87}
\end{align*}
$$

respectively. In addition, the Dirac sea baryon density is given by

$$\rho^{\text{sea}}(x) = \sum_{n \in \Omega} \sum_{\alpha=1}^{g_n} \psi_{n\alpha}^\dagger(x) \psi_{n\alpha}(x) \Theta(-E_n) \tag{88}$$

Similar expressions hold for the scalar and total energy densities. Under the assumption that the zero energy belongs to the gap we can effectively drop the step function and restrict the summation to the lower branch Ω_-. This corresponds to take the limit $\mu \to -\infty$ in the lower branch densities. In this limit each Dirac sea density has a well defined parity under the charge conjugation operation $V \to -V$. The former discussion amounts to the following final equations

$$
\begin{align*}
\rho^{\text{sea}}(x) &= -\rho^+(x, \mu = +\infty) \\
\rho_s^{\text{sea}}(x) &= -\rho_s^+(x, \mu = +\infty) \\
\rho_E^{\text{sea}}(x) &= -\rho_E^+(x, \mu = +\infty) \tag{89}
\end{align*}
$$

Direct use of tables 1, 2 and 3 yields our final result for the WK$\hbar^4$ contributions to the Dirac sea densities. The limit $\mu \to \infty$ corresponds to $k_F \to \infty$ and $x_F \to 1$. The densities can be looked up at table 5 for the case in which no external gauge potential is present. We have checked that the $\hbar^4$ Dirac sea corrections to the baryon density can be written as a total divergence. As a consequence the baryon number remains unchanged. In the particular case of constant scalar field $\Phi = m$ our formulas correspond to the sea corrections of a relativistic atomic system already considered in ref. [16]. We have checked that all our results agree with theirs in this particular case.[5] In addition, we provide the scalar density.

4   Relativistic Nuclear Potentials

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[5] Notice that our definition of $\rho_E - V\rho$ corresponds to the density τ in that reference.

4.1   The σ-ω Model

In this section we analyze the semiclassical expansion in the framework of a mean field approximation to the σ-ω relativistic Lagrangian [1] up to fourth order in $\hbar$. We would like to stress that this model is extremely simple as a realistic description of finite nuclei [2, 3, 4]. However, it adjusts some global properties and reproduces the correct shell structure. Hence we feel it is a convenient starting point to estimate the accuracy of the semiclassical method. To be more specific, we consider the Lagrangian density to be given by (we consider $\hbar = c = 1$)

$$
\mathcal{L} = \overline{\Psi}\left[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)\right]\Psi + \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_s^2\,\phi^2\right) +
$$

$$
- \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\,m_v^2 V_\mu V^\mu\,, \tag{90}
$$

where

$$
F_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu\,. \tag{91}
$$

The field Ψ represents the nucleon field $\Psi = \begin{pmatrix} \psi_P \\ \psi_N \end{pmatrix}$, ϕ stands for the scalar field and V_μ is the vector field. g_v and g_s are dimensionless coupling constants. This Lagrangian may be supplemented with counterterms since it is a renormalizable one. We will always work in the Hartree valence approximation (i.e. no sea approximation), where the scalar and vector fields are considered to be classical and the nucleus wave function is made out of a Slater determinant of A single particle states. We refrain in the present work from making numerical estimates of the Dirac sea corrections. Actually there are no Hartree calculations for finite nuclei which fully include the polarization of the Dirac sea to compare with. In addition one has the vacuum instability problems mentioned in the previous section. The time independent equations defining this approximation in the case of spherical nuclei read

$$
(\nabla^2 - m_s^2)\phi(\mathbf{r}) = -g_s\rho_s(\mathbf{r}) \equiv -g_s\langle\overline{\Psi}\Psi\rangle_{\mathrm{val}}
$$
$$
(\nabla^2 - m_v^2)V_0(\mathbf{r}) = -g_v\rho(\mathbf{r}) \equiv -g_v\langle\overline{\Psi}\gamma_0\Psi\rangle_{\mathrm{val}}
$$
$$
[-i\boldsymbol{\alpha}\cdot\nabla + g_v V_0 + \beta(M - g_s\phi)]\,\psi_{n\pm}(\mathbf{r}) = E_{n\pm}\,\psi_{n\pm}(\mathbf{r})
$$
$$
V_i(\mathbf{r}) = 0,\quad i = 1, 2, 3\,. \tag{92}
$$

Following the standard procedure we solve this set of equations self consistently until convergence is achieved.

Finally the total energy of the system which corresponds to the total nuclear mass can be decomposed as a sum of a fermionic and a bosonic part, as follows

$$
E = E_F + E_B
$$

$$
\begin{aligned}
E_F &= \sum_{\alpha \in \Omega_+} (2j_\alpha + 1)\, E_\alpha \\
E_B &= \frac{1}{2} \int d^3x \left[(\nabla \phi)^2 + m_s \phi^2 \right] - \frac{1}{2} \int d^3x \left[(\nabla V_0)^2 + m_v V_0^2 \right] .
\end{aligned}
\tag{93}
$$

The semiclassical expansion will be applied to the fermionic part of the total energy.

4.2   Fixing of the Parameters

We adjust the parameters of the model in the Hartree valence approximation to nuclear matter properties following [24, 4]. This gives

$$
C_s^2 \equiv g_s^2 \frac{M^2}{m_s^2} = 357.4 \,, \qquad C_v^2 \equiv g_v^2 \frac{M^2}{m_v^2} = 273.8 \,,
\tag{94}
$$

which corresponds to a Fermi momentum $k_F = 1.30\,\text{fm}^{-1}$ and a binding energy per nucleon $B/A = 15.75\,\text{MeV}$. We also fix the nucleon and vector meson masses to their physical values $M = 939$ MeV, $m_v \equiv m_\omega = 783$ MeV. This leaves only one free parameter m_s which we adjust to reproduce the charge mean squared radius of ^{40}Ca (3.482 fm). In summary, the set of parameters to be considered is

$$
\begin{aligned}
M &= 939\text{MeV}, \quad m_s = 449.7\text{MeV}, \quad g_s = \ 9.0547, \\
m_v &= 783 \ \text{ MeV}, \quad g_v = 13.7997.
\end{aligned}
\tag{95}
$$

In what follows we will apply only this parameter set. We emphasize once more that we do not pretend to make a fully realistic description of nuclear properties, but rather to give an idea of how large are the WK$\hbar^4$ corrections to a pure Relativistic Fermi Gas Model.

To have a quantitative idea of their magnitude we might plug the self-consistent scalar and vector potentials into the semiclassical formulas obtained in section (3). As it has been already mentioned, it is necessary to compute the space integrals after an external parameter has been introduced. For numerical computations this is a rather unstable procedure, since in some cases one has to evaluate up to a third derivative with respect to the external parameter. In the next section we will propose a method to circumvent this problem, at the expense of extending the radial dependence of the potentials to a suitable contour in the complex plane encircling the turning point. Strictly speaking one might do so by solving the Hartree mean field equations along this path. In our view this is an extremely complicated task, given the fact that we only want to make a numerical estimate of the corrections. We propose instead an alternative procedure based in adjusting the self consistent

potentials to a given analytical expression which can be then trivially extended to the complex plane. In practice we propose Woods-Saxon potentials of the form

$$\tilde{U} = \frac{U_o}{1 + \exp\left\{\frac{r - R_o}{a_o}\right\}} \qquad (96)$$

to adjust the self consistent potentials $U_V(r) = g_v V(r)$ and $U_\phi(r) = g_s \phi(r)$. To do so we make a least square fit minimizing

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} \left(\tilde{U}(r_i) - U(r_i)\right)^2}{N}}. \qquad (97)$$

The result of such a fit for N = 49 equidistant points between the origin and $R = 12$ fm is given in table 6. To have a more quantitative idea one can look at some nuclear properties such as binding energies[6] and radii as presented in table 7 [34, 35, 36]. As we see the fit is satisfactory for the computed properties.

4.3 Extension of the Potentials to the Complex Plane

For the calculation of the total energy we always use the perturbative scheme, namely we adjust the chemical potential at zeroth order and expand around this point as explained in section 2.5. In general since we deal with two nucleon species we consider a proton and a neutron chemical potential. Furthermore, we also consider the radial dependence of the potentials. The basic integral appearing in the resulting expressions reads

$$I(\lambda) = \int dr\, g(\Phi, V, r)\, \sqrt{(\mu - V(r))^2 - \Phi(r)^2 - \lambda^2}\, \Theta\left(\mu - V(r) - \sqrt{\Phi(r)^2 + \lambda^2}\right). \qquad (98)$$

Other integrals can be obtained from it through successive derivatives with respect to the external parameter λ^2 at the particular value $\lambda = 0$. The potentials we are dealing with, present a turning point at a finite value of the radial coordinate $r_c(\lambda)$, given by the equation

$$\mu - V(r_c(\lambda)) = \sqrt{\Phi(r_c(\lambda))^2 + \lambda^2}. \qquad (99)$$

We assume, as it is often the case, that the function g is analytical in a region of the complex plane which includes the integration interval $[0, r_c(\lambda)]$. We also choose the branch cut of the square root function along the real positive axis

$$\sqrt{z} = |z|^{\frac{1}{2}}\, e^{i\left(\frac{1}{2}\arg z + \pi\right)}, \qquad \arg z\, \epsilon[0, 2\pi[. \qquad (100)$$

---

[6]In no case have we extracted the center of mass motion.

With this choice the integral (98) can be transformed into an integral along the path defined by the lines $C_+ \cup C_-$ depicted in figure 1. Under the assumption that no further singularities occur within the closed contour $C_+ \cup C_2 \cup C_- \cup C_1$ it is clear that the original integral can be evaluated along the external line C_1 of the contour, i.e.

$$
\begin{aligned}
I(\lambda) &= \frac{1}{2} \int_{C_+ \cup C_-} dz \, g(\Phi, V, z) \sqrt{(\mu - V(z))^2 - \Phi(z)^2 - \lambda^2} \\
&= -\frac{1}{2} \int_{C_1} dz \, g(\Phi, V, z) \sqrt{(\mu - V(z))^2 - \Phi(z)^2 - \lambda^2} \, . \qquad (101)
\end{aligned}
$$

In this representation, the turning points never lie on the integration path, and hence the integration and the derivatives with respect to the external parameter λ at $\lambda = 0$ can be interchanged. In this case no explicit distributive contribution appears, provided we keep away from the turning points. It should be mentioned that for quantities which involve logarithms there might appear in principle a different analytical structure. A detailed analysis reveals that also in this case formula (101) remains valid. The discussion above, allows us to readily use the results of tables 1-3 in the radial case literally with the additional modification of removing the step function and understanding the radial coordinate to lie along path C_1 in the complex plane. Finally, it is important to check that no additional singularities appear when the path C_1 is deformed. We do this in practice by comparing our prescription (101) with the integral calculated along the real axis, when no derivative with respect to the external parameter appears. From the point of view of the complex plane, this condition is sufficient since derivatives in the external parameter do not introduce new singular points.

4.4 Numerical Results

We consider the Woods-Saxon potentials described above and apply the contour prescription explained in the previous section. The results are presented in table 8. There we show the numerical values for the total fermionic energy at any given order, together with the exact results (WS) for those potentials as well as the Strutinsky averaged energy (SA). The latter is computed in the cases where it can be easily extracted, i.e. when there are enough bound states over the chemical potential. In no case have we considered center of mass corrections. In general we observe similar trends as in the harmonic potentials case, with the exception that for lighter nuclei the convergence of the semiclassical series seems to be slower. It is not clear whether this behaviour corresponds to that of an asymptotic series. Finally, it is worth mentioning that in the cases where the Strutinsky average is well defined,

the semiclassical expansion seems to converge to a value compatible with the uncertainty inherent in the Strutinsky average. The conclusions drawn for the harmonic potentials seem to hold also in the case of Woods-Saxon potentials; whereas the WK$\hbar^2$ corrections to the Fermi Gas Model provide a substantial improvement to the total energy, this is no the case for the WK$\hbar^4$ corrections, which turn out to be much smaller than pure shell effects.

5   Conclusions and Perspectives

In the present paper we have studied the role of $\hbar^4$ corrections to the extended WK approximation within the relativistic approach to nuclear physics. For this purpose, we have considered the cumulative number and level density of the Dirac operator with scalar and vector potentials as the basic objects to which the semiclassical approximation can be applied. This allows to compute any one-body operator within a semiclassical treatment and more important, reduces the calculational effort considerably. We have found extremely convenient to relate the cumulative number of states to the phase of the determinant of the Dirac operator. Furthermore, we transform the whole Dirac problem into a Schrödinger type one with some additional spinor structure, which makes the even nature of the $\hbar$ expansion more evident. In fact, our methodology reproduces previous $\hbar^2$ calculations [5, 6] rather quickly. We have then obtained explicit expressions for the total energy, scalar and particle densities up to $\hbar^4$-order. We have found that, for central potentials, a precise meaning to the densities can be given by considering them along a path in the complex radial coordinate which starts and finishes at $r = 0$ and encloses the classical turning points. The mean values of such densities produce convergent results, at the expense of extending the scalar and vector potentials to the complex r-plane. We have also shown how the WK$\hbar^4$ corrections for the Dirac sea can be deduced from the valence densities and their explicit form has been given.

To have a quantitative estimate of the WK$\hbar^4$ corrections, we have considered the relativistic σ-ω Lagrangian in the Hartree valence approximation. We have applied a semiclassical expansion to the fermionic part of the total nuclear mass and determined the contribution of the different orders in $\hbar$. As external profiles we have considered a Woods-Saxon fit to the exact self-consistent Hartree solution, allowing the aforementioned extension to the complex plane. We have compared the semiclassical expansion with the exact result and also with the corresponding Strutinsky average in the cases where the latter can be easily extracted. We have found that in those cases the Strutinsky average produces, within its inherent uncertainty, quite accurately the semiclassical WK results up to fourth order. In general we have

found that pure shell effects are far more important than the $\hbar^4$-corrections, but of comparable size as the $\hbar^2$ contributions. Nevertheless, one should say that our statement is strictly true for the total energy which makes out the nuclear mass, but does not necessarily imply that higher $\hbar$ corrections are completely irrelevant for other observables, like e.g. form factors at not too low momentum transfers. It should be also mentioned that our results do not imply either that the $\hbar^4$ order correction to the extended TF approximation has to be small. In fact, as found in ref. [37] in a non-relativistic framework, the WK expansion seems to converge faster than a density functional approach.

Although our method has been applied to the σ-ω Lagrangian, it can be directly generalized to more realistic and sophisticated relativistic Lagrangians including Coulomb interaction, ρ exchange, etc. which correctly reproduce a wealth of nuclear data. We do not expect our results to be substantially modified by such generalization.

Our numerical calculation requires further approximations besides a pure WK expansion. The need to produce convergent results requires an analytical continuation of the potentials to the complex radial coordinate plane. Given the fact that the self-consistent potentials are available on a numerical grid, we have chosen to fit analytical Woods-Saxon expressions to them, allowing a direct extension to the complex plane. One could perform better, by solving the Hartree equations on a given path of the complex r-plane.

In no case have we considered a direct minimization of the semiclassical energy as a functional of the bosonic fields. Besides our wish to keep the problem manageable, we do not think that this is the correct way of proceeding, specially when the $\hbar^4$ corrections are included. These corrections include fourth order derivatives which very strongly influence the behaviour of the system at short distances, require additional boundary conditions not present in the original formulation of the problem and might even spoil the mere existence of solutions. At this point there is a dramatic difference with the $\hbar^2$ corrections. In this case the number and form of the derivatives coincide with those of the pure classical bosonic energy. A systematic treatment of this question for higher order corrections might be achieved following the suggestion of ref. [25, 26] where the solutions are expanded in powers of $\hbar$ around the Thomas-Fermi solution, combined with our complex plane prescription. A detailed study of this problem is left for future research.

In summary, we might conclude by saying that our results suggest a clear and compelling calculational scheme in the semiclassical approach to relativistic nuclear physics. The WK$\hbar^0$ approximation already explains many of the gross features and substantial improvement can be achieved by computing the next to leading order corrections, i.e. $\hbar^2$ in the WK expansion. From there on, the semiclassical expansion

is not only difficult to handle, but also accounts for a tiny fraction of the $\hbar^0$ plus the $\hbar^2$ contributions. The difference between the semiclassical expansion and the exact result is due to pure shell effects which are comparable in magnitude to the $\hbar^2$ contributions and cannot be incorporated within a semiclassical treatment.

6   Acknowledgements

Appendix. Semiclassical Expansion of the Functional Determinant

In this appendix we briefly review a very powerful method to compute a derivative expansion of the functional determinant, which has been first suggested within Quantum Field Theory [27] to compute effective actions in the low energy limit. It only requires the operator inside the determinant to be a differential operator of elliptic type, i.e. of the form

$$P^2 + U(X) \tag{102}$$

where $P_\mu = i\hbar\,\partial_\mu + A_\mu(X)$ is the generalized momentum operator containing in the most general case a non-abelian gauge field A_μ. For completeness we just quote the main steps of the method presented in [27, 28]. $U(X)$ is an operator depending on the position operator X_μ and can have any internal symmetry structure. The following general relations are satisfied

$$
\begin{aligned}
\left[X_\mu, X_\nu\right] &= 0 \\
\left[P_\mu, X_\nu\right] &= i\hbar\,\delta_{\mu\nu} \\
\left[P_\mu, P_\nu\right] &= i\,F_{\mu\nu} \\
\left[P_\mu, U\right] &= i\,\mathcal{D}_\nu U
\end{aligned}
\tag{103}
$$

The Euclidean indices μ and ν run from 1 to D (the Euclidean space dimension). The basic object to be expanded is the determinant of the elliptic operator

$$
\begin{aligned}
W &= \log \mathrm{Det}\left(P^2 + U(X)\right) \\
&= \mathrm{Tr}\,\log(P^2 + U(X)).
\end{aligned}
\tag{104}
$$

In general this is an ultraviolet divergent quantity which in principle requires regularization. However, in the applications we will be considering only the phase of the determinant which turns out to be convergent, and hence no explicit ultraviolet regularization will be introduced. In addition, the determinant is invariant under similarity transformations. A particularly interesting one is given by the translation operator in momentum space, which generates the identity

$$W = \mathrm{Tr}\left(e^{\frac{i}{\hbar}kX} \log(P^2 + U(X)) e^{-\frac{i}{\hbar}kX}\right) = \mathrm{Tr}\,\log\left((P_\mu + k_\mu)^2 + U(X)\right), \tag{105}$$

where k is an arbitrary c-numerical vector. Since this is valid for any momentum k, one can average over all possible values, as follows

$$
\begin{aligned}
W &= \Omega^{-1} \int \frac{d^D k}{(2\pi\hbar)^D}\, \mathrm{Tr}\,\log\left(\Delta^{-1} + 2kP + P^2\right) \\
\Omega &= \int \frac{d^D k}{(2\pi\hbar)^D}\,, \qquad \Delta(X) = \left(k^2 + U(X)\right)^{-1}.
\end{aligned}
\tag{106}
$$

It should be mentioned that the averaging procedure corresponds to project onto the gauge invariant subspace, and hence implicitly assumes a gauge invariant regulator for W. This expression suggests an expansion in the generalized momentum operator P_μ by use of the formula

$$W = \Omega^{-1} \int \frac{d^D k}{(2\pi\hbar)^D} \text{Tr} \left(\log(\Delta^{-1}) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\Delta(2kP + P^2))^n \right). \qquad (107)$$

Although the starting average defines a gauge invariant object, the former expansion does not preserve gauge invariance explicitly, i.e. term by term. The main goal is to restore this invariance in an explicit manner. We will not dwell upon the details which have been described at length elsewhere [28]. The final result can be cast in the form

$$W = \int \frac{d^D k \, d^D x}{(2\pi\hbar)^D} \sum_{n=0}^{\infty} \hbar^{2n} \, W_{2n}(x, k). \qquad (108)$$

For later reference we reproduce in table 9 the general results up to sixth order. Note that the infinite volume factor Ω cancels in the final result, as it should be.

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TABLE CAPTIONS

1. Wigner-Kirkwood expansion of the valence contribution to the baryonic density, $\langle \psi^\dagger(x)\psi(x)\rangle_{\text{val}}$, in the relativistic case up to fourth order in $\hbar$ for $D = 3$ dimensions. The definitions $\epsilon_F = \mu - V$, $k_F = (\epsilon_F^2 - \Phi^2)^{1/2}$, $x_F = \epsilon_F/k_F$ and $l_F = \log((\epsilon_F + k_F)/\Phi)$ have been used. For more details sea main text and eq. (79).

2. The same as table 1 but for the scalar density, $\langle \overline{\psi}(x)\psi(x)\rangle_{\text{val}}$.

3. The same as table 1 but for the valence contribution to the total fermionic energy density, $\langle \psi^\dagger(x)(-i\alpha \cdot \nabla + \beta\Phi(x) + V(x))\psi(x)\rangle_{\text{val}}$.

4. Wigner-Kirkwood expansion for the total energy (in MeV) of a system of A nucleons in relativistic harmonic potentials $V = \Phi - m = \frac{1}{4}m\omega^2 r^2$ where $\hbar\omega = 41A^{-1/3}$. WK_n includes the semiclassical corrections up to $\hbar^n$. NP represents the non-perturbative determination of the chemical potential, whereas P stands for the perturbative determination (see main text). SA is the Strutinsky averaged total energy with Gaussian weight factors. E_{ex} is the exact quantum-mechanical total energy. In all cases the energy mA has been subtracted. The numbers for A, ($A = Z + N$ and $Z = N$) represent the corresponding magic numbers for these potentials.

5. $\text{WK}\hbar^4$-corrections to the Dirac sea baryonic, scalar and total energy densities respectively.

6. Numerical fit of the parameters of the Woods-Saxon Potentials (see eq. (96)) to the self-consistent mean field ϕ, V_0, solutions in the Hartree valence approximation with 49 points. σ (in MeV) is the standard deviation between the Hartree solutions and the Woods-Saxon potentials within the physically relevant region $0 \leq r \leq 12$ fm. U_o is given in MeV, and R_o and a_o are presented in fm.

7. Comparison of the binding energy per nucleon, B/A (MeV), and mean squared charge radius, m.s.c.r. (fm), for several closed shell nuclei. σ-ω means the σ-ω self consistent Hartree solutions with the parameter set specified in the text. WS stands for the best fit of Woods-Saxon potentials to the corresponding self consistent scalar and vector fields. For general information we also quote the experimental values whenever they are accessible.

8. Semiclassical expansion for the total fermionic energy in MeV for several closed shell nuclei. The scalar an vector potentials are of Woods-Saxon type fitted

to the self consistent Hartree valence potentials (see table 6). The chemical potential has been always adjusted within the perturbative method (see section 2.5). WK_n contains the corrections up to order $\hbar^n$. SA stands for the Strutinsky averaged energy and WS represents the quantum mechanical energy. In all cases the rest mass of the constituent nucleons M has been subtracted.

9. Values of the coefficients $W_{2n}(x, k)$ as defined in eq. (108). The following convention has been used:

   (a) Given an object Y_I with $I = (\alpha_1 \ldots \alpha_n)$ a set of Euclidean indices we define $Y_{I\mu}$ to mean $-i[P_\mu, Y_I]$; for instance $\Delta_{\mu\nu} = (-i)^2[P_\nu, [P_\mu, \Delta]]$; $F_{\mu\nu\rho} = -i[P_\mu, F_{\mu\nu}]$.

   (b) Each term in the table means the half-sum of the term itself and its mirror symmetric. The mirror symmetry is defined as $P_\mu \to P_\mu^T = P_\mu$; $\Delta \to \Delta^T = \Delta$ and $YZ \to (YZ)^T = ZY$. For instance $\Delta_\mu^T = -\Delta_\mu$; $F_{\mu\nu}^T = -F_{\mu\nu}$; $F_{\mu\nu\rho}^T = F_{\mu\nu\rho}$.

   (c) Notice the cyclic permutation symmetry.

FIGURE CAPTION

1. Contour in the radial complex plane used to evaluate radial integrals in the classically allowed region (defined by the path C_+), namely $\int_{C_+} = -\frac{1}{2}\int_{C_1}$. The branch cut starts at the radial turning point $r_c(\lambda)$ defined in the text.

$$\rho_0^+ = \frac{k_F^3}{3\,\pi^2\,\hbar^3}\,\Theta(\mu - V - \Phi)$$

$$\rho_2^+ = \frac{1}{24\,\pi^2\,\hbar}\Biggl\{ \qquad \frac{1}{k_F}\left(2 - x_F{}^2\right)(\nabla\Phi)^2 \qquad + \qquad \frac{2}{\Phi}x_F\left(3 - x_F{}^2\right)(\nabla_i\Phi)(\nabla_i V)$$

$$+\frac{1}{k_F}\left(3 - x_F{}^2\right)(\nabla V)^2 \; - \; \frac{2}{k_F}\Phi(\nabla^2\Phi) \; - \; 2\left(x_F + 2l_F\right)(\nabla^2 V)\Biggr\}\,\Theta(\mu - V - \Phi)$$

$$\rho_4^+ = \frac{\hbar}{24\,\pi^2}\Biggl\{ \qquad \frac{1}{10\,k_F{}^3}\,\Phi(\nabla^4\Phi) \qquad + \qquad \frac{1}{10\,\Phi^2}\,x_F\left(-5 + x_F{}^2\right)(\nabla^4 V)$$

$$+\frac{1}{20\,k_F{}^3}\left(-5 + 4\,x_F{}^2\right)(\nabla_i\nabla_j\Phi)^2 \qquad + \qquad \frac{1}{60\,k_F{}^3}\left(-19 + 12\,x_F{}^2\right)(\nabla_i\nabla_j V)^2$$

$$+\frac{1}{30\,\Phi^3}\,x_F\left(45 - 35\,x_F{}^2 + 12\,x_F{}^4\right)(\nabla_i\nabla_j\Phi)(\nabla_i\nabla_j V)$$

$$+\frac{3}{10\,\Phi^3}\,x_F\left(5 - 5\,x_F{}^2 + 2\,x_F{}^4\right)(\nabla_i V)(\nabla_i\nabla^2\Phi)$$

$$+\frac{1}{10\,k_F{}^3}\left(-7 + 6\,x_F{}^2\right)(\nabla_i\Phi)(\nabla_i\nabla^2\Phi) \qquad + \qquad \frac{1}{5\,k_F{}^3}\left(-5 + 3\,x_F{}^2\right)(\nabla_i V)(\nabla_i\nabla^2 V)$$

$$+\frac{1}{5\,\Phi^3}\,x_F\left(15 - 10\,x_F{}^2 + 3\,x_F{}^4\right)(\nabla_i\nabla^2 V)(\nabla_i\Phi)$$

$$+\frac{1}{10\,k_F{}^5}\,\Phi\left(-14 + 15\,x_F{}^2\right)(\nabla_i\Phi)(\nabla_i\nabla_j\Phi)(\nabla_j\Phi)$$

$$+\frac{1}{10\,\Phi^4}\,x_F\left(-75 + 125\,x_F{}^2 - 102\,x_F{}^4 + 30\,x_F{}^6\right)(\nabla_i V)(\nabla_i\nabla_j\Phi)(\nabla_j\Phi)$$

$$+\frac{1}{30\,\Phi^4}\,x_F\left(-135 + 205\,x_F{}^2 - 159\,x_F{}^4 + 45\,x_F{}^6\right)(\nabla_i\Phi)(\nabla_i\nabla_j V)(\nabla_j\Phi)$$

$$+\frac{3}{k_F{}^7}\,\Phi^3\,(\nabla_i V)(\nabla_i\nabla_j V)(\nabla_j\Phi) \qquad + \qquad \frac{1}{80\,k_F{}^5}\left(96 - 280\,x_F{}^2 + 175\,x_F{}^4\right)(\nabla\Phi)^4$$

$$+\frac{1}{60\,\Phi^5}\,x_F\left(900 - 2455\,x_F{}^2 + 3351\,x_F{}^4 - 2145\,x_F{}^6 + 525\,x_F{}^8\right)(\nabla_i\Phi)(\nabla_i V)(\nabla\Phi)^2$$

$$+\frac{1}{8\,k_F{}^5}\left(18 - 55\,x_F{}^2 + 35\,x_F{}^4\right)(\nabla V)^2(\nabla\Phi)^2$$

$$+\frac{1}{12\,\Phi^4}\,x_F\left(-54 + 73\,x_F{}^2 - 54\,x_F{}^4 + 15\,x_F{}^6\right)(\nabla^2 V)(\nabla\Phi)^2$$

$$+\frac{1}{10\,k_F{}^5}\,\Phi\left(-13 + 15\,x_F{}^2\right)(\nabla_i V)(\nabla_i\nabla_j\Phi)(\nabla_j V)$$

$$+\frac{1}{30\,\Phi^4}\,x_F\left(-90 + 185\,x_F{}^2 - 156\,x_F{}^4 + 45\,x_F{}^6\right)(\nabla_i V)(\nabla_i\nabla_j V)(\nabla_j V)$$

$$+\frac{1}{20\,k_F{}^5}\left(89 - 275\,x_F{}^2 + 175\,x_F{}^4\right)(\nabla_i\Phi)(\nabla_i V)(\nabla_j\Phi)(\nabla_j V)$$

$$+\frac{1}{60\,\Phi^5}\,x_F\left(675 - 2270\,x_F{}^2 + 3264\,x_F{}^4 - 2130\,x_F{}^6 + 525\,x_F{}^8\right)(\nabla V)^2(\nabla_i\Phi)(\nabla_i V)$$

$$+\frac{1}{10\,k_F{}^5}\,\Phi\left(-26 + 25\,x_F{}^2\right)(\nabla^2 V)(\nabla_i\Phi)(\nabla_i V)$$

$$+\frac{1}{80\,k_F{}^5}\left(63 - 270\,x_F{}^2 + 175\,x_F{}^4\right)(\nabla V)^4$$

$$+\frac{1}{20\,k_F{}^5}\,\Phi\left(-18 + 25\,x_F{}^2\right)(\nabla\Phi)^2(\nabla^2\Phi)$$

$$+\frac{1}{30\,\Phi^4}\,x_F\left(-135 + 265\,x_F{}^2 - 237\,x_F{}^4 + 75\,x_F{}^6\right)(\nabla_i\Phi)(\nabla_i V)(\nabla^2\Phi)$$

$$+\frac{1}{4\,k_F{}^5}\,\Phi\left(-3 + 5\,x_F{}^2\right)(\nabla V)^2(\nabla^2\Phi) \qquad + \qquad \frac{1}{20\,k_F{}^3}\left(-3 + 5\,x_F{}^2\right)(\nabla^2\Phi)^2$$

$$+\frac{1}{6\,\Phi^3}\,x_F\left(9 - 8\,x_F{}^2 + 3\,x_F{}^4\right)(\nabla^2 V)(\nabla^2\Phi)$$

$$+\frac{1}{60\,\Phi^4}\,x_F\left(-135 + 305\,x_F{}^2 - 261\,x_F{}^4 + 75\,x_F{}^6\right)(\nabla V)^2(\nabla^2 V)$$

$$+\frac{1}{60\,k_F{}^3}\left(-26 + 15\,x_F{}^2\right)(\nabla^2 V)^2 \qquad + \qquad \frac{1}{\hbar^4\,k_F}\,F_{ij}^2\Biggr\}\,\Theta(\mu - V - \Phi)$$

Table 1

$$\rho_{s\,0}^{+} = \frac{1}{2\,\pi^2\,\hbar^3}\left\{\epsilon_F\,k_F\,\Phi - \Phi^3\,l_F\right\}\Theta(\mu - V - \Phi)$$

$$\rho_{s\,2}^{+} = \frac{1}{24\,\pi^2\,\hbar}\left\{-\frac{1}{\Phi}\,x_F\,(2 + x_F{}^2)\,(\nabla\Phi)^2 - \frac{2}{k_F}\,\Phi(\nabla^2 V) - \frac{2}{k_F}\,(2 + x_F{}^2)\,(\nabla_i\Phi)(\nabla_i V)\right.$$

$$\left.+\frac{1}{\Phi}\,x_F\,(1 + x_F{}^2)\,(\nabla V)^2 \qquad + \qquad 2\,(-x_F + 3\,l_F)\,(\nabla^2\Phi)\quad\right\}\Theta(\mu - V - \Phi)$$

$$\rho_{s\,4}^{+} = \frac{\hbar}{24\,\pi^2}\left\{\frac{1}{10\,\Phi^2}\,x_F\,(2 + x_F{}^2)\,(\nabla^4\Phi) \qquad\qquad + \qquad\qquad \frac{1}{10\,k_F{}^3}\,\Phi\,(\nabla^4 V)\right.$$

$$+\frac{1}{60\,\Phi^3}\,x_F\,(-21 - 13\,x_F{}^2 + 12\,x_F{}^4)\,(\nabla_i\nabla_j\Phi)^2 \qquad + \qquad \frac{5}{4\,k_F{}^5}\,\Phi\,x_F{}^2\,(\nabla^2 V)(\nabla\Phi)^2$$

$$+\frac{1}{10\,k_F{}^3}\,(1 + 4\,x_F{}^2)\,(\nabla_i\nabla_j\Phi)(\nabla_i\nabla_j V) \quad + \quad \frac{1}{20\,k_F{}^5}\,\Phi\,(3 + 25\,x_F{}^2)\,(\nabla V)^2(\nabla^2 V)$$

$$+\frac{1}{60\,\Phi^3}\,x_F\,(-9 - 17\,x_F{}^2 + 12\,x_F{}^4)\,(\nabla_i\nabla_j V)^2 \qquad + \qquad \frac{1}{2\,k_F{}^3}\,x_F{}^2\,(\nabla^2 V)(\nabla^2\Phi)$$

$$+\frac{3}{5\,\Phi^3}\,x_F\,(-2 - x_F{}^2 + x_F{}^4)\,(\nabla_i\Phi)(\nabla_i\nabla^2\Phi)$$

$$+\frac{1}{5\,k_F{}^3}\,(2 + 3\,x_F{}^2)\,(\nabla_i V)(\nabla_i\nabla^2\Phi) \qquad + \qquad \frac{1}{10\,k_F{}^3}\,(-1 + 6\,x_F{}^2)\,(\nabla_i\nabla^2 V)(\nabla_i\Phi)$$

$$+\frac{3}{10\,\Phi^3}\,x_F\,(-1 - 3\,x_F{}^2 + 2\,x_F{}^4)\,(\nabla_i V)(\nabla_i\nabla^2 V)$$

$$+\frac{1}{30\,\Phi^4}\,x_F\,(60 + 35\,x_F{}^2 - 96\,x_F{}^4 + 45\,x_F{}^6)\,(\nabla_i\Phi)(\nabla_i\nabla_j\Phi)(\nabla_j\Phi)$$

$$+\frac{3}{5\,k_F{}^5}\,\Phi\,(2 + 5\,x_F{}^2)\,(\nabla_i V)(\nabla_i\nabla_j\Phi)\,(\nabla_j\Phi)$$

$$+\frac{1}{10\,k_F{}^5}\,\Phi\,(4 + 15\,x_F{}^2)\,(\nabla_i\Phi)(\nabla_i\nabla_j V)(\nabla_j\Phi)$$

$$+\frac{1}{10\,\Phi^4}\,x_F\,(15 + 35\,x_F{}^2 - 66\,x_F{}^4 + 30\,x_F{}^6)\,(\nabla_i V)(\nabla_i\nabla_j V)(\nabla_j\Phi)$$

$$+\frac{1}{80\,\Phi^5}\,x_F\,(-160 - 80\,x_F{}^2 + 507\,x_F{}^4 - 530\,x_F{}^6 + 175\,x_F{}^8)\,(\nabla\Phi)^4$$

$$+\frac{1}{4\,k_F{}^5}\,(-8 - 18\,x_F{}^2 + 35\,x_F{}^4)\,(\nabla_i\Phi)(\nabla_i V)(\nabla\Phi)^2$$

$$+\frac{1}{24\,\Phi^5}\,x_F\,(-30 - 85\,x_F{}^2 + 309\,x_F{}^4 - 315\,x_F{}^6 + 105\,x_F{}^8)\,(\nabla V)^2(\nabla\Phi)^2$$

$$+\frac{1}{30\,\Phi^4}\,x_F\,(30 + 40\,x_F{}^2 - 93\,x_F{}^4 + 45\,x_F{}^6)\,(\nabla_i V)(\nabla_i\nabla_j\Phi)(\nabla_j V)$$

$$+\frac{1}{2\,k_F{}^5}\,\Phi\,(1 + 3\,x_F{}^2)\,(\nabla_i V)(\nabla_i\nabla_j V)(\nabla_j V)$$

$$+\frac{1}{60\,\Phi^5}\,x_F\,(-165 - 415\,x_F{}^2 + 1542\,x_F{}^4 - 1575\,x_F{}^6 + 525\,x_F{}^8)\,((\nabla_i\Phi)(\nabla_i V))^2$$

$$+\frac{1}{4\,k_F{}^5}\,(-6 - 17\,x_F{}^2 + 35\,x_F{}^4)\,(\nabla V)^2(\nabla_j\Phi)(\nabla_j V)$$

$$+\frac{1}{30\,\Phi^4}\,x_F{}^3\,(130 - 183\,x_F{}^2 + 75\,x_F{}^4)\,(\nabla^2 V)(\nabla_j\Phi)(\nabla_j V)$$

$$+\frac{1}{240\,\Phi^5}\,x_F\,(-135 - 400\,x_F{}^2 + 1506\,x_F{}^4 - 1560\,x_F{}^6 + 525\,x_F{}^8)\,(\nabla V)^4$$

$$+\frac{1}{60\,\Phi^4}\,x_F\,(120 + 50\,x_F{}^2 - 159\,x_F{}^4 + 75\,x_F{}^6)\,(\nabla\Phi)^2\,(\nabla^2\Phi)$$

$$+\frac{1}{10\,k_F{}^5}\,\Phi\,(11 + 25\,x_F{}^2)\,(\nabla_i\Phi)\,(\nabla_i V)\,(\nabla^2\Phi)$$

$$+\frac{1}{12\,\Phi^4}\,x_F\,(6 + 13\,x_F{}^2 - 30\,x_F{}^4 + 15\,x_F{}^6)\,(\nabla V)^2(\nabla^2\Phi)$$

$$+\frac{1}{60\,\Phi^3}\,x_F\,(-33 - 14\,x_F{}^2 + 15\,x_F{}^4)\,(\nabla^2\Phi)(\nabla^2\Phi)$$

$$\left.+\frac{1}{60\,\Phi^3}\,x_F\,(18 - 31\,x_F{}^2 + 15\,x_F{}^4)\,(\nabla^2 V)^2 \qquad + \qquad \frac{1}{\hbar^4\,\Phi}\,x_F\,F_{ij}^2\right\}\Theta(\mu - V - \Phi)$$

Table 2

$$\rho_{E0}^+ = \frac{1}{8\,\pi^2\,\hbar^3}\left\{\epsilon_F^3\,k_F + \epsilon_F\,k_F^3 - \Phi^4\,l_F\right\}\Theta(\mu - V - \Phi) + \rho_0^+\,V$$

$$\rho_{E2}^+ = \frac{1}{24\,\pi^2\,\hbar}\Bigg\{2\,\Phi\,(-x_F + l_F)\,(\nabla^2\Phi) - 2\,\frac{\Phi^3}{k_F^3}\,(\nabla_i\Phi)(\nabla_i V) - 2\,k_F\,(1 + x_F^2)\,(\nabla^2 V)$$

$$+\,(x_F - x_F^3 - l_F)\,(\nabla\Phi)^2 \;+\; (2x_F - x_F^3 - 2l_F)\,(\nabla V)^2\,\Bigg\}\Theta(\mu - V - \Phi) \;+\; \rho_2^+\,V$$

$$\rho_{E4}^+ = \frac{\hbar}{24\,\pi^2}\Bigg\{ \qquad \frac{1}{10\,\Phi}\,x_F^3\,(\nabla^4\Phi) \qquad + \qquad \frac{1}{10\,k_F}\,(-3 + x_F^2)\,(\nabla^4 V)$$

$$+\frac{1}{60\,\Phi^2}\,x_F^3\,(-23 + 12\,x_F^2)\,(\nabla_i\nabla_j\Phi)^2 + \frac{1}{30\,k_F^3}\,\Phi\,(-7 + 12\,x_F^2)\,(\nabla_i\nabla_j\Phi)\,(\nabla_i\nabla_j V)$$

$$+\frac{1}{20\,\Phi^2}\,x_F^3\,(-9 + 4\,x_F^2)\,(\nabla_i\nabla_j V)^2 \quad + \quad \frac{1}{10\,\Phi^2}\,x_F^3\,(-11 + 6\,x_F^2)\,(\nabla_i\Phi)(\nabla_i\nabla^2\Phi)$$

$$+\frac{1}{10\,k_F^3}\,\Phi\,(-1 + 6\,x_F^2)\,(\nabla_i V)(\nabla_i\nabla^2\Phi) \qquad + \qquad \frac{3}{5\,k_F^5}\,\Phi^3\,(\nabla_i\nabla^2 V)(\nabla_i\Phi)$$

$$+\frac{1}{5\,\Phi^2}\,x_F^3\,(-7 + 3\,x_F^2)\,(\nabla_i V)(\nabla_i\nabla^2 V)$$

$$+\frac{1}{30\,\Phi^3}\,x_F^3\,(100 - 123\,x_F^2 + 45\,x_F^4)\,(\nabla_i\Phi)(\nabla_i\nabla_j\Phi)(\nabla_j\Phi)$$

$$+\frac{1}{10\,k_F^3}\,(1 - 36\,x_F^2 + 30\,x_F^4)\,(\nabla_i V)(\nabla_i\nabla_j\Phi)(\nabla_j\Phi)$$

$$+\frac{1}{6\,k_F^3}\,(1 - 12\,x_F^2 + 9\,x_F^4)\,(\nabla_i\Phi)(\nabla_i\nabla_j V)(\nabla_j\Phi)$$

$$+\frac{1}{5\,\Phi^3}\,x_F^3\,(35 - 42\,x_F^2 + 15\,x_F^4)\,(\nabla_i V)(\nabla_i\nabla_j V)(\nabla_j\Phi)$$

$$+\frac{1}{48\,\Phi^4}\,x_F^3\,(-208 + 444\,x_F^2 - 363\,x_F^4 + 105\,x_F^6)\,(\nabla\Phi)^2(\nabla\Phi)^2$$

$$+\frac{1}{20\,k_F^5}\,\Phi\,(-6 - 165\,x_F^2 + 175\,x_F^4)\,(\nabla_i\Phi)(\nabla_i V)(\nabla\Phi)^2$$

$$+\frac{5}{24\,\Phi^4}\,x_F^3\,(-40 + 87\,x_F^2 - 72\,x_F^4 + 21\,x_F^6)\,(\nabla V)^2(\nabla\Phi)^2$$

$$+\frac{1}{12\,k_F^3}\,(4 - 21\,x_F^2 + 15\,x_F^4)\,(\nabla^2 V)(\nabla\Phi)^2$$

$$+\frac{1}{6\,\Phi^3}\,x_F^3\,(19 - 24\,x_F^2 + 9\,x_F^4)\,(\nabla_i V)(\nabla_i\nabla_j\Phi)(\nabla_j V)$$

$$+\frac{1}{30\,k_F^3}\,(-8 - 57\,x_F^2 + 45\,x_F^4)\,(\nabla_i V)(\nabla_i\nabla_j V)(\nabla_j V)$$

$$+\frac{1}{60\,\Phi^4}\,x_F^3\,(-995 + 2172\,x_F^2 - 1800\,x_F^4 + 525\,x_F^6)\,((\nabla_i\Phi)(\nabla_i V))^2$$

$$+\frac{1}{20\,k_F^5}\,\Phi\,(-19 - 160\,x_F^2 + 175\,x_F^4)\,(\nabla V)^2(\nabla_i\Phi)(\nabla_i V)$$

$$+\frac{1}{10\,\Phi^3}\,x_F^3\,(60 - 71\,x_F^2 + 25\,x_F^4)\,(\nabla^2 V)(\nabla_i\Phi)(\nabla_i V)$$

$$+\frac{1}{80\,\Phi^4}\,x_F^3\,(-285 + 689\,x_F^2 - 595\,x_F^4 + 175\,x_F^6)\,(\nabla V)^4$$

$$+\frac{1}{60\,\Phi^3}\,x_F^3\,(140 - 189\,x_F^2 + 75\,x_F^4)\,(\nabla\Phi)^2(\nabla^2\Phi)$$

$$+\frac{1}{10\,k_F^3}\,(-1 - 24\,x_F^2 + 25\,x_F^4)\,(\nabla_i\Phi)(\nabla_i V)(\nabla^2\Phi)$$

$$+\frac{1}{12\,\Phi^3}\,x_F^3\,(25 - 36\,x_F^2 + 15\,x_F^4)\,(\nabla V)^2(\nabla^2\Phi)$$

$$+\frac{1}{60\,\Phi^2}\,x_F^3\,(-19 + 15\,x_F^2)\,(\nabla^2\Phi)^2 \qquad + \qquad \frac{1}{6\,k_F^3}\,\Phi\,(-1 + 3\,x_F^2)\,(\nabla^2 V)(\nabla^2\Phi)$$

$$+\frac{1}{60\,k_F^3}\,(-19 - 96\,x_F^2 + 75\,x_F^4)\,(\nabla V)^2(\nabla^2 V)$$

$$+\frac{1}{20\,\Phi^2}\,x_F^3\,(-12 + 5\,x_F^2)\,(\nabla^2 V)^2 \;+\; \frac{1}{\hbar^4}\,(x_F - l_F)\,F_{ij}^2\,\Bigg\}\Theta(\mu - V - \Phi) \;+\; \rho_4^+\,V$$

Table 3

A		WK$_0$	WK$_2$	WK$_4$	SA	E$_{ex}$
4	NP:	139.428	162.018	161.104	161.1	153.412
	P:	139.428	160.405	161.066		
16	NP:	557.711	592.013	591.437	591.44	579.979
	P:	557.711	591.011	591.427		
40	NP:	1394.28	1440.21	1439.78	1439.78	1424.52
	P:	1394.28	1439.47	1439.78		
80	NP:	2788.56	2846.08	2845.74	2845.74	2826.66
	P:	2788.56	2845.50	2845.74		
140	NP:	4879.97	4949.07	4948.79	4948.80	4925.89
	P:	4879.97	4948.59	4848.79		
224	NP:	7807.96	7888.63	7888.39	7888.39	7861.67
	P:	7807.96	7888.21	7888.39		

Table 4

$$\rho_4^{\rm sea} = \frac{\hbar}{24\,\pi^2}\Bigg\{ \frac{2}{5}\,\Phi^{-2}(\nabla^4 V) \quad - \quad \frac{11}{15}\,\Phi^{-3}(\nabla_i\nabla_j\Phi)(\nabla_i\nabla_j V) \quad - \quad \frac{3}{5}\,\Phi^{-3}(\nabla_i V)(\nabla_i\nabla^2\Phi)$$

$$-\frac{8}{5}\,\Phi^{-3}(\nabla_i\nabla^2 V)(\nabla_i\Phi) + \frac{11}{5}\,\Phi^{-4}(\nabla_i V)(\nabla_i\nabla_j\Phi)(\nabla_j\Phi) + \frac{22}{15}\,\Phi^{-4}(\nabla_i\Phi)(\nabla_i\nabla_j V)(\nabla_j\Phi)$$

$$-\frac{44}{15}\,\Phi^{-5}(\nabla_i\Phi)(\nabla_i V)(\nabla\Phi)^2 \;+\; \frac{5}{3}\,\Phi^{-4}(\nabla^2 V)(\nabla\Phi)^2 \;+\; \frac{8}{15}\,\Phi^{-4}(\nabla_i V)(\nabla_i\nabla_j V)(\nabla_j V)$$

$$-\frac{16}{15}\,\Phi^{-5}(\nabla V)^2(\nabla_i\Phi)(\nabla_i V) \;+\; \frac{16}{15}\,\Phi^{-4}(\nabla_i\Phi)(\nabla_i V)(\nabla^2\Phi) \;-\; \frac{2}{3}\,\Phi^{-3}(\nabla^2 V)(\nabla^2\Phi)$$

$$+\frac{4}{15}\,\Phi^{-4}(\nabla V)^2(\nabla^2 V)\Bigg\}$$

$$\rho_{s4}^{\rm sea} = \frac{\hbar}{24\,\pi^2}\Bigg\{ -\frac{3}{10}\,\Phi^{-2}(\nabla^4\Phi) \quad + \quad \frac{11}{30}\,\Phi^{-3}(\nabla_i\nabla_j\Phi)^2 \quad + \quad \frac{7}{30}\,\Phi^{-3}(\nabla_i\nabla_j V)^2$$

$$+\frac{6}{5}\,\Phi^{-3}(\nabla_i\Phi)(\nabla_i\nabla^2\Phi) \;+\; \frac{3}{5}\,\Phi^{-3}(\nabla_i V)(\nabla_i\nabla^2 V) \;-\; \frac{22}{15}\,\Phi^{-4}(\nabla_i\Phi)(\nabla_i\nabla_j\Phi)(\nabla_j\Phi)$$

$$-\frac{7}{5}\,\Phi^{-4}(\nabla_i V)(\nabla_i\nabla_j V)(\nabla_j\Phi) \;+\; \frac{11}{10}\,\Phi^{-5}(\nabla\Phi)^4 \;+\; \frac{2}{3}\,\Phi^{-5}(\nabla V)^2(\nabla\Phi)^2$$

$$-\frac{11}{15}\,\Phi^{-4}(\nabla_i V)(\nabla_i\nabla_j\Phi)(\nabla_j V) + \frac{22}{15}\,\Phi^{-5}\left((\nabla_i\Phi)(\nabla_i V)\right)^2 - \frac{11}{15}\,\Phi^{-4}(\nabla^2 V)(\nabla_j\Phi)(\nabla_j V)$$

$$+\frac{4}{15}\,\Phi^{-5}(\nabla V)^4 \;-\; \frac{43}{30}\,\Phi^{-4}(\nabla\Phi)^2\,(\nabla^2\Phi) \;-\; \frac{1}{3}\,\Phi^{-4}(\nabla V)^2(\nabla^2\Phi)$$

$$+\frac{8}{15}\,\Phi^{-3}(\nabla^2\Phi)^2 \qquad - \qquad \frac{1}{30}\,\Phi^{-3}(\nabla^2 V)^2\Bigg\}$$

$$\rho_{E4}^{\rm sea} = \frac{\hbar}{24\,\pi^2}\Bigg\{ -\frac{1}{10}\,\Phi^{-1}(\nabla^4\Phi) \quad + \quad \frac{11}{60}\,\Phi^{-2}(\nabla_i\nabla_j\Phi)^2 \quad + \quad \frac{1}{4}\,\Phi^{-2}(\nabla_i\nabla_j V)^2$$

$$+\frac{1}{2}\,\Phi^{-2}(\nabla_i\Phi)(\nabla_i\nabla^2\Phi) \;+\; \frac{4}{5}\,\Phi^{-2}(\nabla_i V)(\nabla_i\nabla^2 V) \;-\; \frac{11}{15}\,\Phi^{-3}(\nabla_i\Phi)(\nabla_i\nabla_j\Phi)(\nabla_j\Phi)$$

$$-\frac{8}{5}\,\Phi^{-3}(\nabla_i V)(\nabla_i\nabla_j V)(\nabla_j\Phi) \;+\; \frac{11}{24}\,\Phi^{-4}(\nabla\Phi)^4 \;+\; \frac{5}{6}\,\Phi^{-4}(\nabla V)^2(\nabla\Phi)^2$$

$$-\frac{2}{3}\,\Phi^{-3}(\nabla_i V)(\nabla_i\nabla_j\Phi)(\nabla_j V) \qquad + \qquad \frac{49}{30}\,\Phi^{-4}\left((\nabla_i\Phi)(\nabla_i V)\right)^2$$

$$-\frac{7}{5}\,\Phi^{-3}(\nabla^2 V)(\nabla_i\Phi)(\nabla_i V) \qquad + \qquad \frac{1}{5}\,\Phi^{-4}(\nabla V)^4 \qquad - \qquad \frac{13}{30}\,\Phi^{-3}(\nabla\Phi)^2(\nabla^2\Phi)$$

$$-\frac{1}{3}\,\Phi^{-3}(\nabla V)^2(\nabla^2\Phi) \quad + \quad \frac{1}{15}\Phi^{-2}\,(\nabla^2\Phi)^2 \quad + \quad \frac{7}{20}\,\Phi^{-2}(\nabla^2 V)^2\Bigg\} \quad + \quad \rho_4^{\rm sea}\,V$$

Table 5

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$_Z^A$X		U_o	R_o	a_o	σ
$_8^{16}$O	$V:$	343.8	2.630	0.5602	2.33
	$\phi:$	415.4	2.624	0.6156	2.16
$_{20}^{40}$Ca	$V:$	401.5	3.518	0.6831	5.76
	$\phi:$	478.8	3.550	0.7175	5.19
$_{20}^{48}$Ca	$V:$	386.7	3.880	0.6074	5.35
	$\phi:$	464.5	3.902	0.6458	4.52
$_{28}^{56}$Ni	$V:$	379.2	4.175	0.5265	4.97
	$\phi:$	457.9	4.190	0.5698	4.57
$_{40}^{90}$Zr	$V:$	374.0	4.954	0.5752	5.28
	$\phi:$	452.2	4.973	0.6193	5.21
$_{50}^{132}$Sn	$V:$	371.7	5.685	0.5745	1.86
	$\phi:$	449.6	5.712	0.6151	2.88
$_{82}^{208}$Pb	$V:$	377.0	6.634	0.6200	1.99
	$\phi:$	455.0	6.669	0.6571	1.94

Table 6

	σ-ω		WS		Exp.	
$^{A}_{Z}$X	B/A	m.s.c.r.	B/A	m.s.c.r.	B/A	m.s.c.r.
$^{16}_{8}$O	3.33	2.84	3.38	2.83	7.98	2.73
$^{40}_{20}$Ca	6.27	3.48*	6.22	3.48	8.55	3.48
$^{48}_{20}$Ca	6.50	3.47	6.34	3.49	8.67	3.47
$^{56}_{28}$Ni	7.23	3.72	7.00	3.73	8.64	
$^{90}_{40}$Zr	8.35	4.22	8.10	4.21	8.71	4.27
$^{132}_{50}$Sn	8.80	4.60	8.58	4.63	8.36	
$^{208}_{82}$Pb	9.83	5.34	9.59	5.37	7.87	5.50

Table 7

		WK$_0$	WK$_2$	WK$_4$	WS	SA
^{16}O	π, ν	-143.4	-143.9	-143.4	-146.6	
^{40}Ca	π, ν	-467.5	-475.2	-473.9	-476.4	
^{48}Ca	π	-554.1	-561.2	-559.4	-562.0	
	ν	-625.2	-639.0	-636.9	-639.1	
^{56}Ni	π, ν	-735.1	-749.0	-746.5	-751.0	
^{90}Zr	π	-1182	-1196	-1193	-1193	-1193
	ν	-1282	-1303	-1300	-1309	
^{132}Sn	π	-1709	-1722	-1719	-1728	-1720
	ν	-2060	-2097	-2092	-2106	
^{208}Pb	π	-2848	-2868	-2864	-2878	-2866
	ν	-3346	-3395	-3391	-3407	

Table 8

$W_2(x,k) = \frac{k^2}{D}\text{tr}[\ ]$		
Δ_μ^2		

$W_4(x,k) = -2\frac{k^4}{D(D+2)}\text{tr}[\ ]$					
-2	Δ_μ^4	$+$	$(\Delta_\mu\Delta_\nu)^2$	$+2$	$(\Delta\Delta_{\mu\mu})^2$
$-$	$(F_{\mu\nu}\Delta^2)^2$	$-4i$	$\Delta F_{\mu\nu}\Delta\Delta_\mu\Delta_\nu$		

$W_6(x,k) = 32\frac{k^6}{D(D+2)(D+4)}\text{tr}[\ ]$					
$\frac{5}{3}$	Δ_μ^6	$-\frac{1}{2}$	$(\Delta_\mu\Delta_\nu\Delta_\mu)^2$	$+\frac{1}{6}$	$(\Delta_\mu\Delta_\nu\Delta_\rho)^2$
$+\frac{1}{2}$	$(\Delta_\mu^2\Delta_\nu)^2$	$-$	$\Delta_\mu^2(\Delta_\nu\Delta_\rho)^2$	-4	$\Delta\Delta_\mu\Delta_\nu\Delta_\mu\Delta_\rho\Delta_{\rho\nu}$
$+4$	$\Delta\Delta_\mu\Delta_\nu^2\Delta_\rho\Delta_{\rho\mu}$	$+\frac{3}{2}$	$\Delta^2\Delta_{\mu\mu}\Delta_\nu^2\Delta_{\rho\rho}$	-4	$(\Delta\Delta_{\mu\mu})^2\Delta_\nu^2$
-2	$(\Delta\Delta_\mu\Delta_{\nu\mu})^2$	-4	$\Delta\Delta_\mu\Delta_{\nu\mu}\Delta_\nu\Delta\Delta_{\rho\rho}$	-2	$\Delta\Delta_\mu\Delta_\nu\Delta\Delta_{\rho\mu}\Delta_{\rho\nu}$
$+2$	$\Delta\Delta_\mu\Delta_\nu\Delta\Delta_{\rho\nu}\Delta_{\rho\mu}$	$+\frac{5}{2}$	$(\Delta\Delta_\mu\Delta_{\nu\nu})^2$	$-$	$\Delta\Delta_\mu\Delta_{\nu\nu}\Delta\Delta_{\rho\rho}\Delta_\mu$
$+2$	$\Delta\Delta_\mu\Delta_{\nu\nu}\Delta_\mu\Delta\Delta_{\rho\rho}$	$-\frac{2}{3}$	$(\Delta\Delta_{\mu\mu})^3$	$+3$	$\Delta^2\Delta_{\mu\mu}\Delta_\nu\Delta\Delta_{\rho\rho\nu}$
$+\frac{3}{4}$	$(\Delta^2\Delta_{\nu\nu\mu})^2$				
$+2\,i$	$F_{\mu\nu}\Delta\Delta_\mu\Delta_\nu\Delta_\rho^2\Delta$	$+2\,i$	$F_{\mu\nu}\Delta\Delta_\mu\Delta_\rho\Delta_\nu\Delta_\rho\Delta$	$+i$	$F_{\mu\nu}\Delta\Delta_\mu\Delta_\rho^2\Delta_\nu\Delta$
$-i$	$F_{\mu\nu}\Delta\Delta_\rho\Delta_\mu\Delta_\nu\Delta_\rho\Delta$	$+2\,i$	$F_{\mu\nu}\Delta_\rho\Delta\Delta_\mu\Delta_\nu\Delta\Delta_\rho$	$+4\,i$	$F_{\mu\nu}\Delta^2\Delta_\mu\Delta_{\rho\rho}\Delta_\nu\Delta$
$+4\,i$	$F_{\mu\nu}\Delta^2\Delta_{\rho\mu}\Delta_\rho\Delta_\nu\Delta$	$+4\,i$	$F_{\mu\nu}\Delta^2\Delta_\rho\Delta_{\mu\rho}\Delta\Delta_\nu$	$+4\,i$	$F_{\mu\nu}\Delta^2\Delta_{\rho\rho}\Delta\Delta_\mu\Delta_\nu$
$+4\,i$	$F_{\mu\nu}\Delta\Delta_\mu\Delta\Delta_{\rho\rho}\Delta_\nu\Delta$	$+2\,i$	$F_{\mu\nu}\Delta^2\Delta_{\rho\mu}\Delta_{\rho\nu}\Delta^2$	$+4\,i$	$F_{\mu\nu}\Delta^2\Delta_\mu\Delta_{\rho\rho\nu}\Delta^2$
$+4\,i$	$F_{\mu\nu}\Delta^2\Delta_{\rho\rho\mu}\Delta\Delta_\nu\Delta$	$+2\,i$	$\Delta^2F_{\mu\nu\nu}\Delta^2\Delta_\mu\Delta_{\rho\rho}$		
$+\frac{1}{2}$	$(\Delta F_{\mu\nu}\Delta)^2\Delta_\rho^2$	$+$	$F_{\mu\nu}\Delta^2F_{\mu\rho}\Delta\Delta_\nu\Delta_\rho\Delta$	$-$	$F_{\mu\nu}\Delta^2F_{\mu\rho}\Delta\Delta_\rho\Delta_\nu\Delta$
-4	$F_{\mu\nu}\Delta^3F_{\mu\rho}\Delta_\rho\Delta\Delta_\nu$	$+$	$F_{\mu\nu}\Delta^2\Delta_\rho F_{\mu\nu}\Delta_\rho\Delta^2$	-2	$F_{\mu\nu}\Delta^2\Delta_\rho F_{\mu\rho}\Delta^2\Delta_\nu$
$-$	$(\Delta F_{\mu\nu}\Delta\Delta_\mu)^2$	$-\frac{1}{4}$	$(\Delta F_{\mu\nu}\Delta\Delta_\rho)^2$	$+2$	$F_{\mu\nu}\Delta^3F_{\mu\nu}\Delta^2\Delta_{\rho\rho}$
-4	$F_{\mu\nu}\Delta^3F_{\mu\rho\rho}\Delta^2\Delta_\nu$	$-\frac{1}{2}$	$(\Delta^3F_{\mu\nu\nu})^2$		
$-\frac{1}{3}\,i$	$F_{\mu\nu}\Delta^2F_{\mu\rho}\Delta^2F_{\nu\rho}\Delta^2$				

Table 9

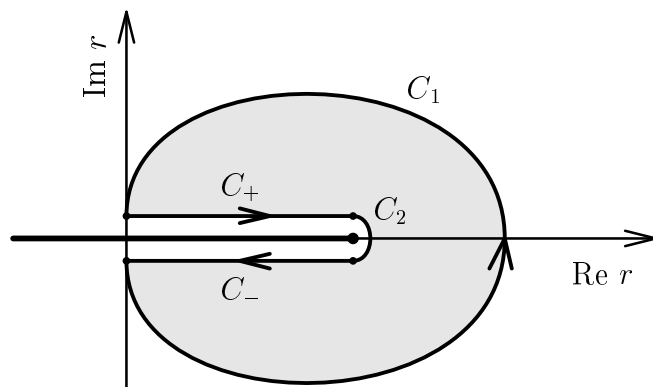This figure "fig1-1.png" is available in "png" format from:

http://arXiv.org/ps/nucl-th/9410025v1

Figure 1